Quantum Theory 2015/16

Tutorial Sheet 8

8.1 Let $\Delta(t,t')$ be the Green function for the classical one-dimensional harmonic oscillator which satisfies

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)\Delta(t,t') = -\delta(t-t') \quad \text{with} \quad \Delta(t_a,t') = \Delta(t_b,t') = 0\,, \tag{1}$$

where $t_a < \{t,\, t'\} < t_b$.

(a) Show that the solutions $x^-(t)$ and $x^+(t)$ of this equation in the regions $(t < t')$ and $(t > t')$ satisfying the boundary conditions $x^-(t_a) = 0$ and $x^+(t_b) = 0$ respectively, are

$$x^-(t) = A\sin\omega(t-t_a) \quad \text{and} \quad x^+(t) = B\sin\omega(t_b - t)$$

where A and B are arbitrary real constants.

Choosing $\Delta(t,t')$ to be continuous at $t = t'$, show by integrating Equation (1) over the infinitesimal range $t'-\epsilon < t < t'+\epsilon$, that

$$\left.\frac{\partial\Delta(t,t')}{\partial t}\right|_{t=t'+\epsilon} - \left.\frac{\partial\Delta(t,t')}{\partial t}\right|_{t=t'-\epsilon} = -1\,.$$

By matching $x^-(t)$ and $x^+(t)$ and the discontinuity in their derivatives at $t = t'$, show that

$$\Delta(t,t') = \frac{1}{\omega\sin\omega T}\Big[\theta(t-t')\sin\omega(t_b-t)\sin\omega(t'-t_a) + \theta(t'-t)\sin\omega(t-t_a)\sin\omega(t_b-t')\Big]$$

Verify by explicit differentiation that this expression for $\Delta(t,t')$ satisfies equation (1).

(b) Show that the classical action for the forced harmonic oscillator of Q2.3 is

$$S[\bar{x},J] = S[\bar{x}_0,0] + \int_{t_a}^{t_b} dt\,\bar{x}_0(t)J(t) - \frac{1}{2m}\int_{t_a}^{t_b} dt\int_{t_a}^{t_b} dt'\,J(t)\Delta(t,t')J(t').$$

Use this to rederive the result of Q5.1, and show that

$$\langle x_b,t_b|T(\hat{x}(t_1)\hat{x}(t_2))|x_a,t_a\rangle_{J=0} = \Big(\bar{x}_0(t_1)\bar{x}_0(t_2) + \tfrac{i\hbar}{m}\Delta(t_1,t_2)\Big)\,\langle x_b,t_b|x_a,t_a\rangle_{J=0}.$$

8.2 Show that

$$\int \frac{d^n x}{(2\pi)^{n/2}}\,\exp(-\tfrac{1}{2}x^TAx + b^Tx) = (\det A)^{-1/2}\exp(\tfrac{1}{2}b^TA^{-1}b),$$

where x and b are real n-dimensional vectors, and A is a real symmetric matrix with positive eigenvalues, so $x^TAx = \sum_{ij} x_iA_{ij}x_j$ etc.

Deduce that if we define

$$\langle x_{i_1}\ldots x_{i_m}\rangle \equiv Z^{-1}\int\frac{d^nx}{(2\pi)^{n/2}}\,x_{i_1}\ldots x_{i_m}\exp(-\tfrac{1}{2}x^TAx),$$

with $Z \equiv \langle 1\rangle$, then if m is odd $\langle x_{i_1}\ldots x_{i_m}\rangle = 0$, while if m is even

$$\langle x_{i_1}\ldots x_{i_m}\rangle = \sum_{\text{all pairs}}(A^{-1})_{j_1k_1}\ldots(A^{-1})_{j_mk_m},$$

where the sum runs over all possible pairings of the m indices. Show that the number of terms in this sum is

$$\frac{m!}{2^{m/2}(m/2)!}$$

(PTO)

8.3 Consider a forced anharmonic oscillator with Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 + Jx - \frac{\lambda}{3!}\,x^3\,.$$

Show that, for $\lambda = 0$,

$$\langle 0, t_b |\, T(\hat{x}(t_1)\,\hat{x}(t_2)\,\hat{x}(t_3))\,|0, t_a\rangle_{J=0} = 0\,,$$

and that to first order in perturbation theory in λ

$$\frac{1}{F_\omega(T)}\,\langle 0, t_b |\, T(\hat{x}(t_1)\,\hat{x}(t_2)\,\hat{x}(t_3))\,|0, t_a\rangle_{J=0} = -\frac{i\lambda}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t\, \prod_{i=1}^{3}\left(\frac{i\hbar}{m}\,\Delta(t, t_i)\right)$$

plus disconnected parts. Draw the relevant Feynman diagram and deduce the Feynman rules (for $J = 0$).

8.4 Confirm explicitly that

$$\Delta_F(t - t') = \frac{1}{2i\omega}(e^{-i\omega(t-t')}\theta(t - t') + e^{i\omega(t-t')}\theta(t' - t))$$

is also a Green function for the classical one dimensional harmonic oscillator, vanishing when $t \to \infty$ or $t \to -\infty$ provided that ω has an negative (infinitesimal) imaginary part ($\omega \to \omega - i\epsilon$). Show that the Fourier transform of $\Delta_F(t)$ is

$$\Delta_F(E) = \frac{1}{E^2 - \omega^2 + i\epsilon}$$

and evaluate the inverse Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dE \frac{e^{-iEt}}{E^2 - \omega^2 + i\epsilon}$$

by contour integration to recover $\Delta_F(t)$, taking care to choose the correct contour. Notice how the $i\epsilon$-prescription automatically gives the Feynman Green function, just as it gave the Schrödinger Green function.

8.5 (a) Show that for small λ

$$I(\lambda) = \int_{-\infty}^{\infty} dx e^{-x^2/2 - \lambda x^4} \sim \sqrt{2\pi}\sum_{k=0}^{\infty}\frac{(-)^k (4k)!}{4^k k!(2k)!}\lambda^k.$$

(b) Use Stirling's formula to show that the coefficient of λ^k in this expansion goes as $(-16k/e)^k$ for large k, and thus that the series diverges.

(c) How could you have guessed that the series would diverge just by looking at the integral? What does this suggest for the convergence (or otherwise) of any perturbation series in quantum mechanics?

[Hint: is $I(\lambda)$ analytic in a neighbourhood of the origin?]