## Section 10: Fourier Transformations and ODEs

### *10. 1. Fourier series*

Recall that we may represent a function defined on some finite interval, say $[-L, L]$, by an infinite sum of plane waves

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp \frac{in\pi x}{L} \ .$$

To calculate the coefficients $C_n$ we use an identity we derived previously

$$\delta_{mn} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \exp ik(m-n) = \int_{-L}^{L} \frac{dy}{2L} \exp \frac{i\pi(m-n)y}{L} \qquad \text{change of variable} \quad y = \frac{kL}{\pi} \tag{1}$$

Then

$$\int_{-L}^{L} \frac{dx}{2L} f(x) \exp -\frac{i\pi nx}{L} = \int_{-L}^{L} \frac{dx}{2L} \sum_{m=-\infty}^{\infty} C_m \exp \frac{i\pi(m-n)x}{L} = \sum_{m=-\infty}^{\infty} C_m \delta_{mn} = C_n$$

We can also construct Fourier sine and cosine series which involve only sines and cosines and therefore have a definite symmetry about the origin. For example if a function $f(x)$ is defined on $[0, L]$ we can choose to extend it to $[-L, L]$ so that it is an odd function $f(-x) = -f(x)$. This extension is very natural if $f(0) = f(L) = 0$ as it introduces no discontinuities when the function is periodically extended to all $x$.

From (1) it is easy to show that

$$\int_{-L}^{L} \frac{dx}{L} \sin\left(\frac{\pi mx}{L}\right) \sin\left(\frac{\pi nx}{L}\right) = \delta_{mn} \quad \text{for} \quad m, n > 0 \ .$$

which can be used to calculate the sine series

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \\ A_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx \end{aligned}$$

Similarly we can construct cosine series.

### *10. 2. Fourier transformation*

You may have been introduced to Fourier transforms (F.T.) in previous courses as a limit of Fourier series as the interval $[-L, L] \to [-\infty, \infty]$. Here we can do better by using the delta function identity we derived in section 6.

We define the F.T of a function $f(x)$ as

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} dx \, f(x) e^{-ikx} \quad \text{often denoted} \quad g(k) \quad \text{or} \quad \tilde{f}(k) \tag{2}$$

Then $f(x)$ may be written as

$$f(x) = \mathcal{F}^{-1}[g(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, g(k) e^{ikx} \ . \tag{3}$$

This can be verifed by using

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}$$

We have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, g(k)e^{ikx} = \int_{-\infty}^{\infty} dk\, \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\, f(x')e^{-ikx'}e^{ikx} = \int_{-\infty}^{\infty} dx'\, f(x')\delta(x-x') = f(x)$$

Note our convention of where to put the $2\pi$ in (2,3), or indeed which integral involves $k$ and which involves $-k$, is immaterial. What we require is that somewhere in the two equations there is a factor $1/2\pi$ and that one integral involves $-k$ and the other $k$.

The F.T. may easily be generalised to higher dimensions e.g. $3d$

$$\mathcal{F}[f(\underline{x})] = g(\underline{k}) = \int_{-\infty}^{\infty} d^3x\, f(\underline{x})e^{-i\underline{k}\cdot\underline{x}} \qquad f(x) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, g(\underline{k})e^{i\underline{k}\cdot\underline{x}} \tag{4}$$

where $\underline{k}$ is a point in 3-d $k$ space.

*Example:* Damped Oscillation

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x/\kappa}\sin k_0 x & x > 0 \end{cases}$$

The Fourier transform is

$$\begin{aligned} g(\omega) &= \int_{-\infty}^{\infty} dx f(x)e^{-ikx} = \int_0^{\infty} dx \frac{e^{-x/\kappa - ikx}}{2i}\left[e^{ik_0x} - e^{-ik_0x}\right] \\ &= \frac{1}{2}\left(\frac{1}{k_0 - k + i/\kappa} + \frac{1}{k + k_0 - i/\kappa}\right) \end{aligned}$$

Note the poles in the frequency spectrum at $k = k_0 \pm i/\kappa$. If $\kappa \gg 1$ (i.e. the wave train becomes very long) then $g(k)$ becomes sharply peaked near $k = \pm k_0$

Let us note some simple general properties: F.T. of a derivative

$$\mathcal{F}[f'(x)] = \int_{-\infty}^{\infty} dx\, f'(x)e^{-ikx} = \left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f(x)(-ik)e^{-ikx} = ik\mathcal{F}[f(x)]$$

where $f(x)$ vanishes at $\pm\infty$ so that $\mathcal{F}[f(x)]$ is well defined. Similarly

$$\mathcal{F}[f''(x)] = -k^2\mathcal{F}[f(x)]$$

Now consider the F.T. of a gaussian, here in $3d$

$$f(\underline{x}) = Me^{-r^2/a^2} \qquad \text{where} \quad M = \left(\frac{2}{\pi a^2}\right)^{3/4}$$

$$g(\underline{k}) = M\int_{-\infty}^{\infty} d^3x\, e^{-r^2/a^2}e^{-i\underline{k}\cdot\underline{x}}$$

(The normalisation $M$ has been chosen so that $|f(\vec{x})|^2$ integrates to 1.) We could perform the integrals in Cartesians giving a product of three identical integrals for $x$, $y$ and $z$. Here

we choose to practise use of spherical polars $d^3x = r^2 \sin\theta dr\, d\theta\, d\phi$

$$\begin{aligned}
g(\underline{k}) &= M2\pi \int_0^\infty dr\, r^2 \int_0^\pi d\theta\, \sin\theta\, e^{-r^2/a^2} e^{-ikr\cos\theta} \\
&= M2\pi \int_0^\infty dr\, r^2 e^{-r^2/a^2} \int_0^\pi d\theta\, \sin\theta\, e^{-ikr\cos\theta} \\
&= M2\pi \int_0^\infty dr\, r^2 e^{-r^2/a^2} \frac{1}{ikr} \left[e^{-ikrcos\theta}\right]_0^\pi = \frac{2\pi M}{ik} \int_{-\infty}^\infty dr\, r e^{-r^2/a^2} e^{ikr} \\
&= \frac{2\pi M}{ik} e^{-k^2a^2/4} \int_{-\infty}^\infty ds(s + ika^2/2) e^{-s^2/a^2} \qquad (r = s + ika^2/2) \\
&= 2\pi M e^{-k^2a^2/4} \frac{a^2}{2} \sqrt{\pi} a = (2\pi a^2)^{3/4} e^{-k^2a^2/4}
\end{aligned}$$

The final result shows that the FT of a gaussian $f(\underline{x})$ of width (standard deviation) $a/\sqrt{2}$ is another gaussian $g(\underline{k})$ of width $\sqrt{2}a$. Thus the widths of the two gaussians are inverse to each other:

$$\Delta x \Delta k = 1\ .$$

This is very closely related to the uncertainty principle in quantum mechanics (take $\underline{p} = \hbar \underline{k}$ where $h$ is Planck's constant).

*10. 3. Convolution*

The convolution of two functions $f_1$, $f_2$ is defined as

$$(f_1 * f_2)(x) = \int_{-\infty}^\infty f_1(y) f_2(x-y) dy\ .$$

The idea is that we are integrating up the product of $f_1$ and $f_2$ where the sum of the arguments is constant. This can arise e.g. as the probability that the sum of two independent random variables with probabilities $f_1$ $f_2$ takes value $x$

Now,

$$\begin{aligned}
\mathcal{F}[f_1 * f_2] &= \int_{-\infty}^\infty dx e^{-ikx} \int_{-\infty}^\infty dy f_1(y) f_2(x-y) \\
&= \int_{-\infty}^\infty dy f_1(y) \int_{-\infty}^\infty dx e^{-ik(x+y)} f_2(x) \\
&= \int_{-\infty}^\infty dy f_1(y) e^{-iky} \int_{-\infty}^\infty dx f_2(x) e^{-ikx} \\
&= \text{const} \times \mathcal{F}[f_1(x)] \mathcal{F}[f_2(x)] \qquad (5)
\end{aligned}$$

where with our '$2\pi$' convention the constant = 1. Thus the F.T. of a convolution factorises into a product of the F.T. of each function.

Importantly there is a converse to this:

$$\mathcal{F}[f_1(x) f_2(x)] = \frac{1}{2\pi} \int_{-\infty}^\infty g_1(k') g_2(k-k') dk' = \frac{1}{2\pi} (g_1 * g_2)(k) \qquad (6)$$

i.e. the F.T. of the product of two functions is equal (up to a multiplicative constant) to the convolution of the two F.T. The proof of this is left to a tutorial.

*10. 4. Solving ODEs using Fourier Transformations*

Method:

1. We seek $y(t)$ as a solution of the ODE + boundary conditions, but a direct solution is often difficult
2. Take F.T. of the ODE: F.T. of $y(x)$ is $\widetilde{y}(k)$ which then a satisfies 'simpler' (usually algebraic) equation
3. Solve this equation
4. Invert $\widetilde{y}(k)$ to obtain $y(x)$ — difficult bit!

*Example: Forced Damped Harmonic Motion*

$$\ddot{y}(t) + 2\kappa\dot{y} + \Omega^2 y(t) = f(t)$$

Here since $t$ is the independent variable we use $\omega$ as Fourier variable instead of $x$. We let $\mathcal{F}[y(t)] = \widetilde{y}(\omega)$ etc and take F.T. of ODE

$$(-\omega^2 + 2i\kappa\omega + \Omega^2)\widetilde{y}(\omega) = \widetilde{f}(\omega) \tag{7}$$

$$\Rightarrow \quad \widetilde{y}(\omega) = \frac{1}{\Omega^2 + 2i\kappa\omega - \omega^2}\widetilde{f}(\omega) \tag{8}$$

Where we have used the properties of F.T. of derivatives. To invert this note that since this is a product of two F.T. we may invoke the convolution theorem (5):

$$y(t) = \int_{-\infty}^{\infty} dt' G(t-t') f(t') \tag{9}$$

$$\text{where} \qquad G(t) = \mathcal{F}^{-1}\left[\frac{1}{\Omega^2 + 2i\kappa\omega - \omega^2}\right]$$

$$= -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)} \qquad \text{with} \quad \omega_\pm = i\kappa \pm \sqrt{\Omega^2 - \kappa^2} \tag{10}$$

$G$ is of course the **Green function** of the problem.

We evaluate the inversion integral for $G(t)$ by closing the contour and using the residue theorem.

For $t > 0$ we close the contour in the upper half plane (because of $e^{i\omega t}$) and $G$ depends on the position of the poles $\omega_\pm$.

$$G(t) = -2\pi i \sum_{\omega_\pm} \text{Res}\left\{\frac{1}{2\pi} \frac{e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)}\right\}$$

$$= -i\left\{\frac{e^{i\omega_+ t}}{(\omega_+ - \omega_-)} + \frac{e^{i\omega_- t}}{(\omega_- - \omega_+)}\right\} \tag{11}$$

For $t < 0$ we close the contour in the lower half plane (because of $e^{-i\omega|t|}$) and there are no residues since $Im\ (\omega_\pm) > 0$. Thus $G(t) = 0$ for $t < 0$.

Now consider the nature of $G(t)$ for $t > 0$.

*i)* $\Omega > \kappa > 0$ *Oscillating (Underdamped) System*

Here the imaginary part of the two simple poles is the same and $\omega_+ - \omega_- = 2\sqrt{\Omega^2 - \kappa^2} > 0$

Thus

$$G(t) = \theta(t) e^{-\kappa t} \frac{\sin(\sqrt{\Omega^2 - \kappa^2}\, t)}{\sqrt{\Omega^2 - \kappa^2}}$$

Actually we could have guessed this as we have previously calculated the F.T. of this function.

*ii)* $\kappa > \Omega > 0$ *Overdamped System*

Here the poles lie on the imaginary axis $\omega_\pm = -i\kappa \pm i\sqrt{\kappa^2 - \Omega^2}$

$$G(t) = \theta(t) e^{-\kappa t} \frac{\sinh(\sqrt{\kappa^2 - \Omega^2}\, t)}{\sqrt{\kappa^2 - \Omega^2}}$$

*iii)* $\kappa = \Omega$ *Critical Damping*

Here there is a single pole of order 2 at $i\kappa$, so for $t > 0$

$$G(t) = 2\pi i \mathrm{Res}\left[ -\frac{1}{2\pi} \frac{e^{i\omega t}}{(\omega - i\kappa)^2} \right] = -i(it) e^{-\kappa t}$$

(or we can obtain the answer by l'Hopital's rule from (11))

$$\text{thus} \qquad G(t) = \theta(t) t e^{-\kappa t}$$

*10. 5. Boundary Conditions*

Actually we have been a little cavalier in our calculations of Green functions so far and we have not explicitly discussed boundary conditions— recall that when we previously calculated Green functions for ODEs it was important which boundary conditions were specified.

In this section we have implicitly taken the boundary condition that the Green function vanish at infinity so that the Fourier transform converges. Also suitably many derivatives must vanish so that all the quantities in the transformed ODE converge. In this way we calculate the **Fundamental Green Function** (which in our previous terminology corresponds to homogeneous boundary conditions at infinity).

For example, in the example of 10.4, in all three cases the Green function vanishes as $t \to \infty$. So in fact (9) is a particular solution of the above problem rather than the general solution. If we want the general solution to the above example when solving (7) we should write

$$\widetilde{y}(\omega) = \frac{1}{\Omega^2 + 2i\kappa\omega - \omega^2} \widetilde{f}(\omega) + 2\pi A \delta(\omega - \omega_+) + 2\pi B \delta(\omega - \omega_-)$$

where $A, B$ are arbitrary constants. (We should generally include the delta functions because with an equation $h(k)g(k) = b(k)$ at the zeros of $h(k)$, $g(k)$ could be a delta function.) Inverting this then gives the general solution

$$y(t) = \int_{-\infty}^{\infty} dt' G(t - t') f(t') + A e^{i\omega_+ t} + B e^{i\omega_- t} \,. \tag{12}$$

*10. 6. Causality*

As a reminder we interpret the equation

$$y_p(t) = \int_{-\infty}^{\infty} dt' G(t-t') f(t')$$

as the integral of the *stimulus* $f$ at time $t' \times$ the *response* $G(t,t')$ at time $t$. In the example $G(t,t') = 0$ for $t < t'$ i.e. there is no response in the past ($t$) to a stimulus in the future ($t'$) i.e. the system is *causal*

Actually the final point (causality) is quite subtle. Note that in the limit $\kappa \to 0$ of our example (no dissipation) the poles $\omega_\pm$ approach the real axis from above. Thus for any small amount of damping the poles will be in the upper half plane giving $G(t,t') = 0$ for $t < t'$. But if we had strictly no dissipation then the poles would lie won the real axis and it would not be clear how to include them in the contour integral. It would not be clear that $G(t,t') = 0$ for $t < t'$ — we would have to impose causality rather than it emerging from the equations.

Actually one fix to this problem is to go back to section 6 and note that strictly we should use the integral representation

$$\delta(x) = \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{dk}{2\pi}\, e^{ikx} \qquad \epsilon > 0\ .$$

i.e. our $k$ integral would end up just below the real axis thanks to $\epsilon$ but this is a bit fiddly so we won't pursue this route.

We will return to the matter of causality later on when we deal with partial differential equations.

*10. 7. Fourier Sine and Cosine Transforms*

For differential equations where the range of $x$ (or $t$) is $[0,\infty]$ it is useful to introduce sine/cosine transforms defined by

$$g(k) = 2\int_0^\infty dx\, f(x) \begin{cases} \sin(xk) \\ \cos(xk) \end{cases} \qquad f(x) = \frac{1}{\pi}\int_0^\infty dk\, g(k) \begin{cases} \sin(xk) \\ \cos(xk) \end{cases}$$

Which one is appropriate depends on the boundary condition at $x = 0$: if $y(0) = 0$ then a sine transform of $y$ is appropriate whereas if $y'(0) = 0$ a cosine transform is appropriate.