METHODS OF MATHEMATICAL PHYSICS

Series expansions of ODE; generating functions

Tutorial Sheet 5

K: key question – explores core material
R: review question – an invitation to consolidate
C: challenge question – going beyond the basic framework of the course
S: standard question – general fitness training!

5.1 **Second order ODEs of physics** [k]

Find the positions and nature of the singularities of the following differential equations

$$\begin{aligned}
x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) &= 0\,, && \text{(Bessel's equation)}\\
y''(x) - 2xy'(x) + 2\lambda y(x) &= 0\,, && \text{(Hermite's equation)}\\
xy''(x) + (1-x)y'(x) + \lambda y(x) &= 0\,, && \text{(Laguerre's equation)}\\
(1-x^2)y''(x) - xy'(x) + \lambda y(x) &= 0\,, && \text{(Chebyshev's equation)}\\
(1-x^2)^2y''(x) - 2x(1-x^2)y'(x) + \left\{\lambda(1-x^2) - \mu\right\} y(x) &= 0\,, && \text{(Associated Legendre equation)}
\end{aligned}$$

5.2 **Euler's Equation** [s] Find two real, linearly-independent solutions to Euler's equation

$$x^2y''(x) + xy'(x) + y(x) = 0$$

by making a series expansion about $x = 0$. Verify that the Wronskian of your solution is a constant times x^{-1}

5.3 **Legendre Polynomials** [s] Write down Legendre's equation

$$(1-x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0$$

in terms of the variable $z = 1 - x$ and obtain a solution in terms of a power series in z.

Show that the series diverges as $z \to 2$ but that if $\lambda = n(n+1)$ where $n = 0, 1, 2 \ldots$, the solution is a polynomial in z.

Obtain these polynomials for $n = 0, 1$ and 2 and derive a linearly-independent solution in the $n = 0$ case.

5.4 **Integral related to generating function for Legendre polynomials** [s]

Consider the integral

$$I(\epsilon) = \int_0^{2\pi} \frac{d\theta}{1 + \epsilon\cos\theta} \qquad |\epsilon| < 1$$

By letting $z = e^{i\theta}$ write it as a contour integral around the unit cricle.

Show by the residue theorem that

$$I(\epsilon) = \frac{2\pi}{\sqrt{1-\epsilon^2}}$$

5.5 **Solving recurrence relations using a generating function** [s]

Consider functions $f_n(x)$, where n is an integer, defined by the recurrence relations

$$\begin{aligned}
(n+1)f_{n+1}(x) &= xf_n(x) - f_{n+2}(x)\\
f_n'(x) &= f_{n-1}(x)
\end{aligned}$$

Calculate the generating function $G(x,t) = \sum_{n=-\infty}^{\infty} f_n(x) t^n$ and show that

$$f_0(x) = Constant \times \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

5.6 **Hermite polynomials [s]**

The Hermite polynomials $H_n(x)$ where $n = 0, 1, 2\ldots$ have the following generating function

$$G(x,h) \equiv \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!} = e^{2hx-h^2}$$

(i) By taking derivatives wrt h and x respectively, find the recurrence relation relating H_{n-1}, H_n, H_{n+1} and the recurrence relation relating H'_n and H_{n-1}

(ii) Use Cauchy's integral formula and the generating function to obtain an integral representation of $H_n(x)$

(iii)* Use the integral representation to evaluate

$$\int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) dx$$

Ans: 0 for n odd

$\sqrt{2\pi}\dfrac{n!}{(n/2)!}$ for n even

5.7 **Asymptotic expansion around $x = \infty$ [r]**

If $y(x)$ satisfies Bessel's equation

$$x^2 y''(x) + xy'(x) + \left[x^2 - n^2\right] y(x) = 0 \quad 0 < x < \infty$$

show that $u(x) = x^{1/2} y(x)$ satisfies

$$u'' + \left[1 - b/x^2\right] u(x) = 0 \quad \text{where} \quad b = (n^2 - 1/4) \ .$$

Develop an asymptotic (large x) expansion of $u(x)$ of the form

$$u = \mathrm{e}^{\pm ix} \sum_{m=0}^{\infty} a_m x^{-m}$$

and show that

$$a_{m+1} = \pm \frac{m(m+1) - b}{2i(m+1)} a_m$$

5.8 **Behaviour near singularities [k]**

If $y(x)$ satisfies the equation

$$x^2 y''(x) + 2xy'(x) + \left[\lambda x - l(l+1) - \frac{x^2}{4}\right] y(x) = 0 \quad 0 < x < \infty$$

find a function $v(x)$ such that $u(x) = y(x)/v(x)$ satisfies an equation of the form

$$u'' + Q(x) u(x) = 0 \ .$$

Hence investigate the behaviour of $y(x)$ in the neighbourhood of the singularites in the original equation (i.e. determine leading behaviours of $u(x)$ near $x = 0$ and $x = \infty$).