EM 3 Section 17: Summary of EM in media; boundary conditions on fields

17. 1. Effect of Magnetic Materials on Inductance

First we have to finish off our description of magnetism with a look at how inductance is affected by magnetisation currents

Example: conducting core in solenoid "A long solenoid of n turns per unit length, length ℓ and cross sectional area $\mathcal{A}$ is filled with ferrite, in which $\underline{M}$ obeys $\underline{M} = \chi_m \underline{H}$ where $\chi_m = 900$. Find the self inductance L."

Recall the definition $L = \Phi_B / I$ this stems from Faraday's law MIII, and is therefore **unchanged** by media. Ampère's law in the static situation $\partial \underline{D}/\partial t = 0$ becomes

$$\begin{aligned} \underline{\nabla} \times \underline{H} &= \underline{J}_f + \underline{0} \\ \Rightarrow \oint \underline{H} \cdot \mathrm{d}\underline{l} &= \int \underline{J}_f \cdot \mathrm{d}\underline{S} = n\ell I \end{aligned}$$

in integral form where I is the usual conduction current. Now note the symmetry: $\underline{H}$ is axial within the solenoid and vanishes outside for large ℓ. Taking a loop as shown in figure, $H = nI$, so $\underline{M}$ is

Figure 1: Solenoid with conducting core: Amperian loop

axial; magnitude $M = \chi_m n I$

Then B also must be axial:

$$\begin{aligned} B &= \mu_0(H+M) = (\chi_m + 1)\mu_0 n I \\ \Rightarrow \Phi_B &= n\mathcal{A}LB = (\chi_m+1)\mu_0 n^2 \mathcal{A} L I \\ \Rightarrow L &= \Phi_B/I = (\chi_m+1)\mu_0 n^2 \mathcal{A}\ell \end{aligned}$$

Thus L is 901 times larger than in vacuum (vacuum case: $\chi_m = 0$). For a ferromagnetic material there is a *very large increase* in self inductance.

On the other hand for diamagnetic/paramagnetic materials there is a small decrease/increase in the self-inductance.

For ferromagnetic materials the energy stored in an inductor increases by a large factor $\mu_r \approx 10^3 - 10^6$: See section 17.3 for energy stored in fields

17. 2. Electromagnetism with media: summary

Maxwell's equations in macroscopic form read

$$\underline{\nabla} \cdot \underline{D} = \rho_f \quad (1)$$

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 && (2) \\ \underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} && (3) \\ \underline{\nabla} \times \underline{H} &= \underline{J}_f + \frac{\partial \underline{D}}{\partial t} && (4) \end{aligned}$$

Definitions of $\underline{D}, \underline{H}$ are

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} \qquad \underline{B} = \mu_0(\underline{H} + \underline{M}) \quad (5)$$

Relations for LIH Media

$$\begin{aligned} \underline{P} &= \chi_E \varepsilon_0 \underline{E} & \underline{M} &= \chi_m \underline{H} && (6) \\ \underline{D} &= \varepsilon_0 \varepsilon_r \underline{E} \equiv \varepsilon \underline{E} & \underline{B} &= \mu_0 \mu_r \underline{H} \equiv \mu \underline{H} && (7) \end{aligned}$$

where $\varepsilon_r = 1 + \chi_E \qquad \mu_r = 1 + \chi_m$

17. 3. Energy densities and Poynting Vector

Recall that $\underline{E} \cdot \underline{J}_f$ is the power delivered per unit volume so the *energy density* u obeys

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \underline{E} \cdot \underline{J}_f \quad (8)$$

Now use modified MIV to express

$$\underline{E} \cdot \underline{J}_f = \underline{E} \cdot (\underline{\nabla} \times \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Furthermore we can use a product rule from lecture 1 to write

$$\begin{aligned} \underline{E} \cdot \underline{J} &= \underline{H} \cdot (\underline{\nabla} \times \underline{E}) - \underline{\nabla} \cdot (\underline{E} \times \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} \\ &= -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{\nabla} \cdot (\underline{E} \times \underline{H}) - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} \\ &= -\frac{\partial}{\partial t}\left(\frac{1}{2}\underline{E} \cdot \underline{D} + \frac{1}{2}\underline{B} \cdot \underline{H}\right) - \underline{\nabla} \cdot (\underline{E} \times \underline{H}) \end{aligned}$$

provided that $\underline{E} \cdot \dot{\underline{D}} = \dot{\underline{E}} \cdot \underline{D}$ and $\underline{B} \cdot \dot{\underline{H}} = \dot{\underline{B}} \cdot \underline{H}$ which is true for linear static media. Then integrating over a volume V of the medium and using the divergence theorem on the second term as usual, we obtain from (8) for the total energy (c.f. section 14)

$$\boxed{\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(\frac{1}{2}\underline{E} \cdot \underline{D} + \frac{1}{2}\underline{B} \cdot \underline{H}\right) \mathrm{d}V - \oint_S (\underline{E} \times \underline{H}) \cdot \mathrm{d}\underline{S}} \quad (9)$$

From the first term we identify the electric and magnetic energy densities as

$$u_M = \frac{1}{2}\underline{B} \cdot \underline{H} \qquad u_E = \frac{1}{2}\underline{E} \cdot \underline{D} \quad (10)$$

and from the second term we identify the Poynting vector as

$$\underline{S} = \underline{E} \times \underline{H} \tag{11}$$

17. 4. Boundary Matching Problems

There are often have sharp interfaces between media. These boundaries acquire nonzero values of σ_P surface polarization charge and $\underline{j}_{mag}$ surface magnetisation current

In keeping with use of MI-MIV in macroscopic form, we want to avoid considering these, and think about **free** charges and currents only ...

1. First condition (from $\underline{\nabla} \cdot \underline{D} = \rho_f$): Divergence theorem:

Figure 2: Gaussian surface for deriving continuity conditions on normal components (similar to Griffiths Fig 2.36)

$$\underline{\nabla} \cdot \underline{D} = \rho_f \qquad \Rightarrow \qquad \oint \underline{D} \cdot \mathrm{d}\underline{S} = (Q_f)_{enclosed}$$

Apply to small pillbox or "patch", vector area $\mathrm{d}\underline{S} = \underline{\hat{n}}\mathrm{d}S$

$$(\underline{D}_2 - \underline{D}_1) \cdot \underline{\hat{n}}\mathrm{d}S = \sigma_f \,\mathrm{d}S$$

surface density of FREE charges only. In the absence of free surface charges D_{normal} is continuous. We can also write this as

$$(\underline{D}_2 - \underline{D}_1) \cdot \underline{\hat{n}} = \sigma_f$$

2. Second condition (from $\underline{\nabla} \cdot \underline{B} = 0$):

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad \Rightarrow \qquad \int \underline{B} \cdot \mathrm{d}\underline{S} = 0$$

Apply to small Gaussian pill box (or "patch")

$$\int \underline{B} \cdot \mathrm{d}\underline{S} = (\underline{B}_2 - \underline{B}_1) \cdot \underline{\hat{n}}\mathrm{d}S = 0$$

Therefore B_{normal} is continuous. This is completely general.

3. Third condition (from $\underline{\nabla} \times \underline{E} = -\partial \underline{B}/\partial t$): $\underline{\hat{t}}$ = unit tangent satisfies $\underline{\hat{t}} \cdot \underline{\hat{n}} = 0$; we take a rectangular loop straddling the interface length ℓ height h

$$\oint \underline{E} \cdot \mathrm{d}\underline{l} = (\underline{E}_1 - \underline{E}_2) \cdot \underline{\hat{t}}\, l \qquad = -\frac{\partial}{\partial t}\Phi_B$$

Figure 3: Amperian loop for deriving continuity conditions on tangential components (similar to Griffiths Fig 2.37)

Unless $\underline{B}$ is infinite, the magnetic flux cutting the loop $\Phi_B \to 0$ as $h \to 0$

$$\Rightarrow (\underline{E}_1 - \underline{E}_2) \cdot \underline{\hat{t}} = 0$$

but $\underline{\hat{t}}$ is arbitrary within plane of the surface: $\underline{E}_{tangential}$ is continuous is completely general as it stands. **N.B.** this is **two** conditions in 3D

4. Fourth condition ($\underline{\nabla} \times \underline{H} = \underline{J}_f + \partial \underline{D}/\partial t$):

$\underline{j}_f$ = free surface current / unit area

$$\oint \underline{H} \cdot \mathrm{d}\underline{l} = \underline{j}_f \cdot \underline{\hat{s}}\,\ell + \frac{\partial \underline{D}}{\partial t} \cdot \underline{\hat{s}}\,\ell h$$

where $\underline{\hat{s}} = \underline{\hat{t}} \times \underline{\hat{n}}$ = unit vector $\perp$ to Ampèrian loop

Now take $h \to 0$: last term vanishes

$$\oint \underline{H} \cdot \mathrm{d}\underline{l} = (\underline{H}_1 - \underline{H}_2) \cdot \underline{\hat{t}}\,\ell = \underline{j}_f \cdot \underline{\hat{s}}\,\ell$$

In the absence of free surface currents $\underline{H}_{tangential}$ is continuous

The general form is rarely needed and may be written in several equivalent ways:

$$\begin{aligned}(\underline{H}_1 - \underline{H}_2) \cdot \underline{\hat{t}} &= \underline{j}_f \cdot \underline{\hat{s}} \\ (\underline{H}_2^{tang} - \underline{H}_1^{tang}) &= \underline{j}_f \times \underline{\hat{n}} \\ (\underline{H}_2 - \underline{H}_1) \times \underline{\hat{n}} &= -\underline{j}_f\end{aligned}$$

Summary of the continuity conditions

1.	D_n	continuous	if $\sigma_f = 0$
2.	B_n	continuous	always
3.	$\underline{E}_t$	continuous	always
4.	$\underline{H}_t$	continuous	if $\underline{j}_f = \underline{0}$

These are key results and you should know the derivations.

Problems with nonzero σ_f or $\underline{j}_f$ are uncommon but for these:

$(\underline{D}_2 - \underline{D}_1) \cdot \underline{\hat{n}} = \sigma_f$ replaces 1

$(\underline{H}_2^{tang} - \underline{H}_1^{tang}) = \underline{j}_f \times \underline{\hat{n}}$ replaces 4