EM 3 Section 2: Revision of Electrostatics

2. 1. Charge Density

At the microscopic level charge is a discrete property of elementary particles.
The fundamental charge of an electron is $-e$, where $e = 1.6 \times 10^{-19}$C.

The charge of a proton is $+e$: $q_p + q_e < 10^{-21}e$. Antimatter has the opposite charge to matter: $q_p + q_{\bar{p}} < 10^{-8}e$. $\Rightarrow$ **there is no charge in a vacuum!**

Classical electromagnetism deals with macroscopic charge distributions.
These are defined by a **charge density**, ρ with units Cm^{-3}:

$$\rho(\underline{r}) = [N_p(\underline{r}) - N_e(\underline{r})]e$$

where N are the number densities of protons and electrons.

The total charge in a volume V is obtained by integration over the volume:

$$\boxed{Q_V = \int_V \rho(\underline{r})\mathrm{d}V} \tag{1}$$

where $\mathrm{d}V$ is a small element of volume $dxdydz$ (sometimes we use $\mathrm{d}\tau$ and sometimes d^3r).

Line charges have a charge density λ with units Cm^{-1}. Surface charges have a charge density σ with units Cm^{-2}. Again the total charge can be obtained by integration:

$$Q_A = \int_A \sigma dS \qquad Q_L = \int_L \lambda dl \tag{2}$$

2. 2. Point charges and δ-function

In electrostatic problems it is common to introduce **point charges** at a particular position $\underline{r}\,'$. These are represented by a delta function:

$$\boxed{\rho(\underline{r}) = Q\delta(\underline{r} - \underline{r}\,')} \tag{3}$$

where:

$$\int_V \delta(\underline{r} - \underline{r}\,') = 1 \quad \text{if } \underline{r}\,' \text{ in } V \qquad \int_V \delta(\underline{r} - \underline{r}\,') = 0 \quad \text{otherwise} \tag{4}$$

N.B. Here we are using the three dimensional delta-function, in Cartesians

$$\delta(\underline{r} - \underline{r}\,') = \delta(x - x')\delta(y - y')\delta(z - z') \tag{5}$$

for this reason one sometimes writes $\delta^3(\underline{r} - \underline{r}\,')$

2. 3. Coulomb's Law

The force between two point charges is given by

$$\boxed{\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\underline{r}}} \tag{6}$$

where $\hat{\underline{r}}$ is a unit vector indicating that the force acts along the line connecting the two charges.

The constant ϵ_0 is known as **the permittivity of free space.**

$$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{CN}^{-1}\mathrm{m}^{-2} \tag{7}$$

A quite accurate and easily remembered number is:

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \mathrm{Nm}^2\mathrm{C}^{-1} \tag{8}$$

We'll take Coulomb's law as the empirical starting point for electrostatics. The inverse square dependence on the separation of the charges is measured to an accuracy of 2 ± 10^{-16} using experiments based on the original experiment by Cavendish (Duffin P.31).

Coulomb forces must be added as vectors using the principle of superposition. Thus the force on a point charge q at $\underline{r}$ due to a charge distribution $\rho(\underline{r}')$ is the sum of all charge elements $\rho(\underline{r}')\mathrm{d}^3r'$ at positions $\underline{r}'$

$$\underline{F} = \frac{q}{4\pi\epsilon_0}\int \frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}\rho(\underline{r}')\mathrm{d}^3r' \tag{9}$$

Example: Force due to a Line Charge

As an example consider the force of a line charge λ on a point charge Q. This can be obtained from the sum of the contributions:

$$\underline{\mathrm{d}F} = \frac{Q\lambda\underline{\mathrm{d}l}}{4\pi\epsilon_0 r^2}\hat{\underline{r}}$$

The components *parallel* to the line charge cancel, so we have to sum the contributions *perpendicular* to the line which are $dF\cos\theta$. This yields

$$F_\perp = \int_{-L/2}^{L/2} \frac{Q\lambda\mathrm{d}l}{4\pi\epsilon_0}\frac{\cos\theta}{(l^2 + a^2)}$$

This integral is best solved by transforming it into an integral over $d\theta$ rather than dl i.e. we make the substitution:

$$l = a\tan\theta \qquad dl = a\sec^2\theta d\theta \qquad r = (l^2 + a^2)^{1/2} = a\sec\theta$$

Figure 1: diagram of integrating over line charge elements $\underline{\mathrm{d}l}$ at angle θ to point charge

$$dF_\perp = \frac{Q\lambda}{4\pi\epsilon_0 a}\int_{-\theta_0}^{\theta_0}\cos\theta d\theta = \frac{Q\lambda}{4\pi\epsilon_0 a}2\sin\theta_0$$

where $L/2 = a\tan\theta_0$ and $\sin\theta_0 = L/2\,(a^2 + L^2/4)^{-1/2}$ (see diagram). Thus

$$F_\perp = \frac{Q\lambda L}{4\pi\epsilon_0 a(a^2 + L^2/4)^{1/2}} \tag{10}$$

where L is the length of the line charge, a is the distance of Q from the line.

We now look at different limits of (10). In the limit of an infinite line charge $L \to \infty$:

$$F_\perp = \frac{Q\lambda}{2\pi\epsilon_0 a} \tag{11}$$

Note that the force due to a line charge falls off with distance like $1/a$

In the "far-field" limit where $a \gg L$

$$F_\perp \simeq \frac{Q\lambda L}{4\pi\epsilon_0 a^2} \tag{12}$$

This is the same as the force due to a point charge $q = \lambda L$ *at the origin*

2. 4. Electric Field

The force that a point charge q experiences is written

$$\boxed{\underline{F} = q\underline{E}} \tag{13}$$

which defines the **electric field:** the electric field the force per unit charge experienced by a small static test charge, q. The electric field is a vector and has units of of NC^{-1} or more usually Vm^{-1}:

Thus comparing with Coulomb's force law we see that the electric field at $\underline{r}$ due to a point charge q at the origin is

$$\boxed{\underline{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\underline{r}}} \tag{14}$$

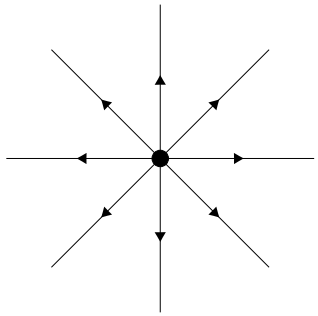

Figure 2: Field lines of electric field emanating from a +ve point charge

which is also known as Coulomb's law.

A positive (negative) point charge is a source (sink) for $\underline{E}$.

In a field line diagram field lines begin at positive charges (or infinity) and end at negative charges (or infinity). The density of field lines indicates the strength of the field.

2. 5. Gauss' law for $\underline{E}$

$$\boxed{\oint_A \underline{E}\cdot\underline{\mathrm{d}S} = \frac{Q}{\epsilon_0}} \tag{15}$$

where A is any closed surface the surface, $\underline{\mathrm{d}S}$ is a vector normal to a surface element, Q is the total charge enclosed by the surface. $\underline{E}(\underline{r})\cdot\underline{\mathrm{d}S}$ is the Electric flux through the surface area element at point $\underline{r}$. The left hand side is often written as

$$\Phi_E = \oint_A \underline{E}\cdot\underline{\mathrm{d}S}$$

which is the total flux of the electric field out of the surface.

Thus *the total Electric flux through any closed surface is proportional to the charge enclosed* (not on how it is distributed)

Simple example of Gauss' law

First let us recover Coulomb's law. Consider a point charge $+q$ at the origin. Take the surface as a sphere of radius r. Now by *symmetry* the field must point radially outwards Thus $\underline{E} = E_r\underline{e}_r$ where E_r has no angular dependence. Then the integral over spherical polar coordinates simplifies considerably

$$\oint_A \underline{E}\cdot\underline{\mathrm{d}S} = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta r^2\sin\theta E_r = 4\pi r^2 E_r$$

and we obtain from Gauss' law $E_r = q/4\pi\epsilon_0 r^2$ which is Coulomb's law.

Aside Although here we have simply stated Gauss' law as fundamental it is actually a *consequence* of the Divergence theorem and Coulomb's Law (14).

Now consider a charge density within the sphere i.e. an insulating sphere with a uniform charge density ρ. Again by symmetry the electric field is radial, i.e. $E_\theta = E_\phi = 0$ everywhere. Again using a spherical closed surface of radius r to calculate the electric field E_r:

$$E_r 4\pi r^2 = \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r^3$$

Inside the sphere ($r < a$):

$$E_r = \frac{\rho}{\epsilon_0}\frac{r}{3}$$

At the surface of the sphere ($r = a$):

$$E_r = \frac{\rho}{\epsilon_0}\frac{a}{3} = \frac{3Q}{4\pi\epsilon_0 a^3}\frac{a}{3} = \frac{Q}{4\pi\epsilon_0 a^2}$$

Outside the sphere ($r > a$):

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

This is the same field as for a point charge Q at the centre of the sphere!

2. 6. Electrostatic Potential

Consider Coulomb's law (14) and the result $\underline{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{\underline{r}}$. We can then write

$$\underline{E} = -\underline{\nabla}\left(\frac{q}{4\pi\epsilon_0 r}\right)$$

and identify the **Electrostatic Potential** at $\underline{r}$ due to a point charge at $\underline{r}'$

$$\boxed{V(\underline{r}) = \frac{q}{4\pi\epsilon_0|\underline{r} - \underline{r}'|}} \tag{16}$$

N.B. Sometimes the symbol ϕ is used for electrostatic potential when there is a possible calsh of notation with V for volume.

Now due to superposition we can integrate Coulomb's law to get the Electric field for any charge distribution and similarly superposition holds for the potential which is then given by e.g. for a continuous charge distribution by

$$\boxed{V(\underline{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\underline{r}')\mathrm{d}^3 r'}{|\underline{r} - \underline{r}'|}} \tag{17}$$

Moreover, we can invoke the important theorem of section 1.7 which implies that due to the existence of the potential V, *static* electric fields generally obey

1. $\underline{\nabla} \times \underline{E} = 0$
2. $\underline{E} = -\underline{\nabla}V$
3. the line integral of the field $\int_A^B \underline{E}\cdot\underline{\mathrm{d}l} = (V_A - V_B)$ is independent of path from A to B
4. a consequence is $\oint_C \underline{E}\cdot\underline{\mathrm{d}l} = 0$ for any closed curve C

N.B this holds only for *static* fields as we shall see later.

N.B. The potential V is only defined up to a constant which may be chosen according to convenience. Often we choose $V = 0$ at $r \to \infty$.

3. gives us the **work done** to move a test charge from A to B

$$\boxed{W_{AB} = -q\int_A^B \underline{E}\cdot\underline{\mathrm{d}l} = q[V_B - V_A]} \tag{18}$$

Note that the work done is defined as the work done by moving the charge *against* the direction of force—hence the minus sign. The potential difference $V_{AB} = V_A - V_B$ is then the energy required to move a test charge between two points A and B, in units of Volts, V=JC^{-1}:

$$V_{AB} = \frac{W_{AB}}{q} \tag{19}$$

If we take A at infinity and $V_A = 0$ the work done to move the charge from infinity to B is the **potential energy** of the test charge at B

$$\boxed{U = qV_B} \tag{20}$$

Warning Beware of confusing Electrostatic potential V and potential energy U

An **equipotential** is a surface connecting points in space which have the same electrostatic potential. By definition $r \to \infty$ is an equipotential with $V(\infty) = 0$.
2. tells us that *The electric field is always perpendicular to an equipotential.*