EM 3 Section 6: Electrostatic Energy and Capacitors

6. 1. Electrostatic Energy of a general charge distribution

Here we provide a proof that the electrostatic energy *density*: (energy per unit volume)

$$\boxed{u_E = \frac{1}{2}\varepsilon_0|\underline{E}|^2} \tag{1}$$

is a completely general result for any electric field.

An assembly of $n-1$ point charges at positions $\underline{r}_j$ gives a potential at $\underline{r}$ (here the subscript j labels the charge):

$$\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\underline{r}-\underline{r}_j|}$$

N.B. Here we use ϕ for potential to avoid confusion between potential and volume V.

Bringing up another charge q_n (the nth one) from infinity to point $\underline{r}_n$ requires work:

$$W_n = q_n\phi(\underline{r}_n) = \frac{q_n}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\underline{r}_n-\underline{r}_j|}$$

i.e. for charge 1, the work is zero since there is no potential yet, for charge 2 the work is $q_2\times$ potential due to charge 1, for charge 3 the work is $q_3\times$ potential due to charges 1 charge 2 etc

So the total energy U_E of the charges, which is equal to the total work required to assemble all n charges, is given by summing over $i = 1,\cdots,n$

$$U_E = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_iq_j}{|\underline{r}_i-\underline{r}_j|} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1(j\neq i)}^{n}\frac{q_iq_j}{|\underline{r}_i-\underline{r}_j|}$$

Make sure you understand the factor of $1/2$ which appears in the second equality when we allow both sums to go over all charges.

We can write the final equality as

$$\boxed{U_E = \frac{1}{2}\sum_i q_i\phi_i} \tag{2}$$

where $\phi_i = \sum_{j\neq i}\frac{q_j}{4\pi\varepsilon_0|\underline{r}_i-\underline{r}_j|}$ is the potential at $\underline{r}_i$ due to the *other* charges j

This can be generalized in the limit of a continuous charge distribution to an integral over the charge density ρ:

$$\boxed{U_E = \frac{1}{2}\int\rho(\underline{r})\phi(\underline{r})\mathrm{d}V} \tag{3}$$

It turns out that we can write this integral in another way. First recall

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_0}\,.$$

Then (3) becomes

$$U_E = \frac{\varepsilon_0}{2}\int\phi(\underline{\nabla}\cdot\underline{E})\,\mathrm{d}V \tag{4}$$

Now use a product rule from section 1 to write

$$\begin{aligned}\phi(\underline{\nabla}\cdot\underline{E}) &= \underline{\nabla}\cdot(\phi\underline{E}) - (\underline{\nabla}\phi)\cdot\underline{E}\\ &= \underline{\nabla}\cdot(\phi\underline{E}) + |\underline{E}|^2\end{aligned}$$

whence (4) yields two integrals. The first can be rewritten using the divergence theorem

$$\int_V\underline{\nabla}\cdot(\phi\underline{E})\mathrm{d}V = \oint_S(\phi\underline{E})\cdot\underline{\mathrm{d}S}$$

Then taking V as *all space* and the boundary S at infinity where we have boundary conditions $\phi(\infty) = 0$ and $\underline{E} = -\underline{\nabla}\phi = 0$ which means that this integral is zero. Therefore the final result comes from the second integral:

$$U_E = \frac{\varepsilon_0}{2}\int_{\text{all space}}|\underline{E}|^2\mathrm{d}V \tag{5}$$

and from this we get (1) for the energy density at a point.

Warning: The general result (5) was derived for a continuous charge distribution since we began from (3). When there are point charges we have to be careful with *self-energy* contributions which should be excluded from the integral (3), otherwise they lead to divergences.

Equation (3) leads us to think that Electrostatic energy lies in the charge distribution whereas from (5) we might infer that the energy is stored in the electrostatic field. Which picture is correct? In fact these interpretations are tautologous.

A final thing to note is that since (5) is quadratic in the field strength we do not have superposition of energy density.

6. 2. Capacitors

A **capacitor** is formed when two neighbouring conducting bodies (any shape) have equal and opposite surface charges. Suppose we have two conductors one with charge Q and the other with charge $-Q$. Since V is constant on each conductor the potential difference between the two is $V = V_1 - V_2$. In general to actually find $V(\underline{r})$ and $\underline{E}$ can be difficult (need to solve Poisson's equation between conductors). However there will be a unique well-defined value of the capacitance defined as the ratio of the charge on each body to the potential difference between the bodies:

$$\boxed{C = \frac{Q}{V_d}} \tag{6}$$

Capacitance is measured in **Farads** = Coulombs/Volt.

A capacitor is basically a device which stores electrostatic energy by charging up.

Figure 1: Diagram of Parallel Plate Capacitor *Griffiths fig 2.52*

Parallel Plate Capacitor

Two parallel plates of area A have a separation d. They carry surface charges of $+\sigma$ and $-\sigma$, which are all on the *inner surface* of the plates because of the attractive force between the charges on the two plates. The normal to the plate is taken in $\underline{e}_z$ direction (positive up).

To obtain the Electric field we use Gauss' Law. First take a pillbox that straddles the *inner surface* of the upper plate say. Now inside the conducting plate $\underline{E} = 0$. Therefore between the two plates the field normal to the inner surface of the upper plate is

$$\underline{E}_{inside} = \left(\frac{(+\sigma)}{\varepsilon_0}\right)(-\underline{e}_z) = -\frac{\sigma}{\varepsilon_0}\underline{e}_z$$

Now take a pillbox that straddles the *outer surface* of the upper plate since there is no charge on the upper surface and since inside the plate $\underline{E}$ is zero we must have that $\underline{E} = 0$ at the outer surface.

$$\underline{E}_{outside} = 0 \qquad \underline{E}_{inside} = -\frac{\sigma}{\varepsilon_0}\underline{e}_z$$

Now

$$E_z = -\frac{\partial V}{\partial z} \quad \Rightarrow \quad V = \frac{\sigma z}{\varepsilon_0}$$

and the potential difference between the plates is:

$$V_d = \frac{Qd}{A\varepsilon_0} \tag{7}$$

$$C = \frac{Q}{V_d} = \frac{A\varepsilon_0}{d} \tag{8}$$

Note that C is a purely geometric property of the plates!

6. 3. Electrostatic Energy

Let us compare the energy of the charge distribution in the capacitor using the two formulas (3,5) derived in the last section.

First use (3): The integral simplifies to a sum of two contributions from the upper plate which has charge Q and potential ϕ_1 and the lower plate which has charge $-Q$ and potential ϕ_2. Thus

$$U = \frac{Q}{2}(\phi_1 - \phi_2) = \frac{QV_d}{2} = \frac{CV_d^2}{2} = \frac{Q^2}{2C} \tag{9}$$

where we have used the definition of capacitance (6).

On the other hand we can integrate over the electric field, which is constant between the plates, using (5).

$$U_E = \frac{\varepsilon_0}{2}\int |\underline{E}|^2 \mathrm{d}V = \frac{\varepsilon_0}{2}Ad\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \frac{d}{2\varepsilon_0}\frac{Q^2}{A}$$

and using the definition of capacitance the result (8)

$$U_E = \frac{Q^2}{2C}$$

6. 4. *Finite size disc capacitors

So far we have assumed the capacitor plates are effectively infinite. In this case the electric field between the two plates was uniform *When should you worry about the finite size of capacitor plates?*

For a *finite-size* capacitor it is possible that there are edge effects where the field can bulge out of the capacitor and also non-uniformity of the field within the capacitor. To get a feeling for when such effects become important let us compute the potential and field due to a *finite-size* disc.

For a finite size disc of charge in the x–y plane, carrying a surface charge density σ, we perform a two-dimensional integral over the charge distribution to obtain the potential at a height z along the axis of the disc

$$V = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma \mathrm{d}S}{(x^2+y^2+z^2)^{1/2}}$$

This integral is easy when we use plane polar co-ordinates for the disc $\mathrm{d}S = \rho \mathrm{d}\rho \mathrm{d}\phi$ where $\rho^2 = x^2 + y^2$ and

$$\begin{aligned} V(z) &= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}\rho\frac{\rho}{(\rho^2+z^2)^{1/2}} \\ &= \frac{\sigma}{4\pi\varepsilon_0}2\pi\left[(\rho^2+z^2)^{1/2}\right]_{\rho=0}^{R} = \frac{\sigma}{2\varepsilon_0}\left[(R^2+z^2)^{1/2} - z\right] \end{aligned}$$

Now the z component of the field is

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left[\frac{z}{(R^2+z^2)^{1/2}} - 1\right]$$

and we see that there is a z dependence so the field is not uniform.

The limit $R \ll z$ corresponds to a point charge:

$$\underline{E}(R \ll z) = \frac{\sigma}{2\varepsilon_0}\left(1 - (1+\frac{R^2}{z^2})^{-1/2}\right)\underline{e}_z \simeq \frac{\sigma R^2}{4\varepsilon_0 z^2}\underline{e}_z = \frac{Q}{4\pi\varepsilon_0 z^2}\underline{e}_z$$

where $Q = \sigma\pi R^2$ is the total charge. The limit $R \gg z$ gives the uniform field of an infinite plane of charge:

$$\underline{E}(R \gg z) = \frac{\sigma}{2\varepsilon_0}\underline{e}_z \tag{10}$$

Only when $R \approx z$ do you need to consider the finite size of the disc—edge effects for a capacitor are limited to the regions where the gap distance is comparable to the distance to the edges of the plates.