EM 3 Section 8: Divergence and Curl of $\underline{B}$; Gauss and Ampère's laws

8. 1. Divergence of $\underline{B}$ and Gauss' Law for Magnetic Fields

We can write the Biot-Savart Law for $\underline{B}$ due to a bulk current density using the expression for $\underline{\nabla}(1/r)$ as

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int_V \underline{J}(\underline{r}') \times \frac{\widehat{(\underline{r}-\underline{r}')}}{|\underline{r}-\underline{r}'|^2}\mathrm{d}V' = -\frac{\mu_0}{4\pi}\int_V \underline{J}(\underline{r}') \times \underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\mathrm{d}V' \tag{1}$$

Now since $\underline{\nabla}$ is with respect to the $\underline{r}$ coordinates and $\underline{J}(\underline{r}')$ depends on $\underline{r}'$ we find

$$\underline{\nabla}\cdot\left(\underline{J}(\underline{r}') \times \underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\right) = \underline{J}(\underline{r}')\cdot\left(\underline{\nabla}\times\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\right) = 0$$

where the last equality follows since 'curlgrad =0'

Therefore

$$\boxed{\underline{\nabla}\cdot\underline{B} = 0} \tag{2}$$

This remarkable result is **the second fundamental law of electromag (Maxwell II)**

A magnetic field has no divergence which is a mathematical statement that there are no magnetic monopoles

This means that there are *no point sources* of magnetic field lines, instead the magnetic fields form *closed loops* round conductors where current flows.

Now the divergence theorem states that

$$\int_V \underline{\nabla}\cdot\underline{B}\,\mathrm{d}V = \oint_A \underline{B}\cdot\underline{\mathrm{d}S}$$

Thus the net magnetic flux through any closed surface A must be zero

$$\boxed{\oint_A \underline{B}\cdot\underline{\mathrm{d}S} = 0} \tag{3}$$

which is sometimes referred to as Gauss' law for magnetic fields.

8. 2. Magnetic Dipoles

Since there are no magnetic monoples we should identify what is the equivalent of an electric dipole i.e. a magnetic dipole. It turns out this is a **current loop**. Consider a circular current loop radius a carrying steady current I in the clockwise direction with axis in the $\underline{e}_z$ direction

We consider the contribution to the magnetic field at $\underline{r}$ *along the axis of the loop* due to the current element $\mathrm{d}\underline{I}(\underline{r}')$ at $\underline{r}'$ using the Biot-Savart law

$$\underline{\mathrm{d}B}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{\underline{\mathrm{d}I}(\underline{r}') \times \widehat{(\underline{r}-\underline{r}')}}{|\underline{r}-\underline{r}'|^2} \tag{4}$$

Figure 1: Simple current loop with axis along z axis

We choose coordinates so that $\underline{r} = z\underline{e}_z$, $\underline{r}' = a\underline{e}_\rho{}'$, $\underline{\mathrm{d}I} = I\mathrm{d}l\underline{e}_\phi{}'$ and $|\underline{r}-\underline{r}'| = (z^2+a^2)^{1/2}$. Then

$$\underline{\mathrm{d}I}(\underline{r}') \times (\underline{r}-\underline{r}') = I\mathrm{d}l(z\underline{e}_\rho{}' + a\underline{e}_z)$$

Now we see that the $\underline{\mathrm{d}B}$ is not along $\underline{e}_z$ but when we integrate around the current loop the perpendicular components cancel. Therefore we consider

$$\mathrm{d}B_z = \frac{\mu_0 I a \mathrm{d}l}{4\pi(z^2+a^2)^{3/2}}$$

Note that this does not depend on the angle ϕ' around the ring therefore when we integrate over dl we simply get a factor $2\pi a$ and

$$B_z = \frac{\mu_0 \mathrm{I} a^2}{2(a^2+z^2)^{3/2}} \tag{5}$$

At the centre of the loop ($z=0$):

$$B_z = \frac{\mu_0 \mathrm{I}}{2a}$$

At a large distance from the loop ($z \gg a$):

$$B_z \simeq \frac{\mu_0 \mathrm{I} a^2}{2z^3}$$

To extend this calculation to an arbitrary position $\underline{r}$ (at all angles θ relative to the axis of the loop) is tedious, but it can be shown that the field of a loop is a magnetic dipole field i.e. in the far field limit of $r \gg a$ one finds

$$\underline{B}_{\mathrm{dip}}(\underline{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\underline{m}\cdot\hat{\underline{r}})\hat{\underline{r}} - \underline{m}\right] \tag{6}$$

where the magnetic dipole moment, $\underline{m}$, is the product of the current and the vector area of the loop:

$$\underline{m} = I\pi a^2\underline{e}_z = IA\underline{e}_z \tag{7}$$

In fact this result holds for a small current loop of *any* shape and with magnetic dipole moment, $\underline{m}$, defined as

$$\boxed{\underline{m} = I\underline{A} = I\int \underline{\mathrm{d}S}} \tag{8}$$

where $\underline{A}$ is the vector area of the loop

Figure 2: Sketch of ideal and physical magnetic dipole lines *Griffiths fig 5.55*

The (ideal) magnetic dipole field has the same form as the (ideal) electric dipole field:

$$B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3} \qquad B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3} \tag{9}$$

However the ‘physical’ versions of electric and magnetic dipole look a bit different
It can be shown (Griffiths 6.1) that: an external magnetic field creates a torque on a magnetic dipole:

$$\underline{T} = \underline{m} \times \underline{B}_{\text{ext}} \tag{10}$$

and the potential energy of the dipole in the field is:

$$U = -\underline{m} \cdot \underline{B}_{\text{ext}} \tag{11}$$

So one can think of a compass needle (magnetic moment along the needle) aligning with the Earth’s magnetic field.

One could think of a magnetic dipole being composed of two monopoles (the ‘Gilbert Model’) and this gives the correct results for torque and energy (see *Griffiths Chapter 6*). However this picture is basically wrong as the fundamental difference between a magnetic dipole and an electric dipole is that it is impossible to separate the N and S poles of a bar magnet for example.

8. 3. What has this got to do with everyday magnets?

You might wonder what current loops have got to do with ordinary bar or fridge magnets. The point is that in most atoms the electrons orbiting an atomic nucleus act as current loops, so atoms can have magnetic dipole moments. Moreover even an electron has ‘spin’ which generates a magnetic moment.

8. 4. Curl of $\underline{B}$ and Ampère’s Law

Consider again the Biot-Savart law in form (1) and take the curl

$$\underline{\nabla} \times \underline{B}(\underline{r}) = -\frac{\mu_0}{4\pi} \int_V \underline{\nabla} \times \left(\underline{J}(\underline{r}') \times \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \right) \mathrm{d}V' \tag{12}$$

where the curl is with respect $\underline{r}$ coordinates so can be taken inside the integral which is over $\underline{r}'$ coordinates. Now using a product rule from lecture 1 and remembering that $\underline{J}(\underline{r}')$ does not depend on the co-ordinates of $\underline{r}$

$$\begin{aligned} \underline{\nabla} \times \left(\underline{J}(\underline{r}') \times \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \right) &= \nabla^2 \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \underline{J}(\underline{r}') - (\underline{J}(\underline{r}') \cdot \underline{\nabla}) \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \\ &= -4\pi\delta(\underline{r} - \underline{r}')\underline{J}(\underline{r}') - (\underline{J}(\underline{r}') \cdot \underline{\nabla}) \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \end{aligned} \tag{13}$$

where we have used the now familiar result $\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\underline{r})$. When we insert (13) back into the integral (12), the second term can be shown to give zero since it can be written as a ‘boundary term’ at infinity which vanishes (see tutorial). The first term in (13) yields

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}} \tag{14}$$

This is another fundamental law of electromagnetism i.e. Maxwell IV
The curl of a magnetic field around an axis is proportional to the component of the current density along the axis.

To obtain an integral form of Ampère’s law we use Stokes’ theorem:

$$\oint_C \underline{B} \cdot \mathrm{d}\underline{l} = \int_S (\underline{\nabla} \times \underline{B}) \cdot \mathrm{d}\underline{S}$$

Thus (14) becomes when we integrate over an open surface S bounded by closed loop C

$$\int_S (\underline{\nabla} \times \underline{B}) \cdot \mathrm{d}\underline{S} = \oint_C \underline{B} \cdot \mathrm{d}\underline{l} = \mu_0 \int_S \underline{J} \cdot \mathrm{d}\underline{S}$$

The integral of the magnetic field round a closed loop is related to the total current flowing across the surface enclosed by the loop:

$$\boxed{\oint_C \underline{B} \cdot \mathrm{d}\underline{l} = \mu_0 \mathrm{I} = \mu_0 \int_S \underline{J} \cdot \mathrm{d}\underline{S}} \tag{15}$$

In a similar way to Gauss’ law in electrostatics, Ampère’s law is very useful for calculating magnetic fields when there is a high degree of symmetry to the problem.

Example: An infinite wire of finitie radius a carries a uniform current density, $\underline{J}$.
Outside the wire at radial distance ρ:

$$\begin{aligned} B_\phi 2\pi\rho &= \mu_0 \int_S \underline{J} \cdot \mathrm{d}\underline{S} = \mu_0 \mathrm{I} \\ B_\phi &= \frac{\mu_0 \mathrm{I}}{2\pi\rho} \end{aligned} \tag{16}$$

The field outside the wire drops off with distance as $1/\rho$. This is a much easier derivation than integrating the Biot-Savart law

Now consider the field inside the wire:

$$\begin{aligned} B_\phi 2\pi\rho &= \mu_0 \int_S \underline{J} \cdot \underline{\mathrm{d}S} = \mu_0 J \pi \rho^2 = \mu_0 \mathrm{I} \frac{\rho^2}{a^2} \\ B_\phi &= \frac{\mu_0 \mathrm{I} \rho}{2\pi a^2} \end{aligned} \tag{17}$$

The field inside the wire increases with radius.