Junior Honours

Electromagnetism **Problem Sheet 10**

Waves in Media

HAND-IN DEADLINE: you must bring your solution to Q10.3 to the teaching office before 4pm Friday 29th March

You are strongly advised to work through the preceding questions, obtaining help from tutors where necessary, **before** attempting the hand-in question.

The code beside each question has the following significance:

- **K**: key question – explores core material
- **R**: review question – an invitation to consolidate
- **C**: challenge question – going beyond the basic framework of the course
- **S**: standard question – general fitness training!

10.1 **Clausius-Mosotti Equation [C]**

A sphere of linear dielectric with permittivity $\varepsilon = \varepsilon_r\varepsilon_0$ of radius a is prepared to have a fixed uniform polarisation vector $\underline{P}$.

(i) Take $\underline{P}$ to be in the $\underline{e}_z$ direction. Write down the surface charge on the sphere.

(ii) Now compute the *resulting* electric field. Use the ansatx of a field in $\underline{e}_z$ direction inside and a dipole ansatz field outside.

Explain why this corresponds to the potential

$$\begin{aligned} V &= A\frac{\cos\theta}{r^2} \qquad \text{for} \quad r > a \\ V &= Br\cos\theta \qquad \text{for} \quad r < a \end{aligned}$$

Use the continuity conditions on the tangential and normal components of $\underline{E} = -\underline{\nabla}V$ at the boundary to show that

$$\underline{E} = -\frac{1}{3\varepsilon_0}\underline{P}$$

Warning: Since $\underline{P}$ is held fixed in this example $\underline{D}$ and $\underline{E}$ are in opposite directions inside the sphere and the relation $\underline{D} = \varepsilon\underline{E}$ does not hold

(iii) Now consider the same sphere in an applied external electric field $\underline{E}_0$. Inside the sphere the total electric field is by superposition

$$\underline{E} = \underline{E}_0 - \frac{1}{3\varepsilon_0}\underline{P}$$

Fix $\underline{P}$ self-consistently using the relation

$$\underline{P} = \varepsilon_0\chi_E\underline{E} = (\varepsilon - \varepsilon_0)\underline{E}\ .$$

You should obtain

$$\underline{P} = \frac{3\varepsilon_0(\varepsilon-\varepsilon_0)}{\varepsilon + 2\varepsilon_0}\underline{E}$$

(iv) Now use the definitions of the polarisation vector

$$\underline{P} = n\langle \underline{p}_{atom}\rangle$$

where n is the number of atoms/molecules per unit volume and $\langle \underline{p}_{atom}\rangle$ is the atomic polarisation and

$$\underline{p}_{atom} = \alpha\underline{E}_0$$

to find an expression for α the atomic polarizability.

10.2 **Incommunicado? [S]**

The electrical conductivity of seawater is $4\Omega^{-1}\text{m}^{-1}$ and its permittivity at radio frequencies is $\varepsilon_r\varepsilon_0 = 1.2\times 10^{-11}\text{Fm}^{-1}$. (Its relative permeability is $\mu_r = 1.0$.)

Is the sea water a good or poor conductor?

What is the approximate skin depth of radio waves with wavelength 3000m? Comment on the relevance for communication with submarines.

10.3♣ **Skin deep**

(i) For a medium of conductivity σ derive the following equation for the B field:

$$\nabla^2\underline{B} = \mu\varepsilon\frac{\partial^2\underline{B}}{\partial t^2} + \mu\sigma\frac{\partial\underline{B}}{\partial t}$$

where $\mu = \mu_r\mu_0$ and $\varepsilon = \varepsilon_r\varepsilon_0$. [Hint: Take the curl of Ampère-Maxwell.] [4]

(ii) Show that this gives the following dispersion relation for a plane wave $\underline{B} = \underline{\tilde{B}}_0 e^{i(\tilde{k}z-\omega t)}$

$$\tilde{k} = (\mu\varepsilon\omega^2 + i\mu\sigma\omega)^{1/2} \tag{1}$$

Explain the criterion based on the ratio

$$\frac{\sigma}{\varepsilon\omega}$$

for whether a medium is a poor or good conducting.[3]

(iii) Define the skin depth δ and *by expanding* (1) appropriately, obtain the limiting expressions (given in lecture 19) for the skin depth in poor and good conductors. **N.B. you are asked here for a differnt derivation to that in the lecture notes** [5]

(iv) Show using the third Maxwell equation that the complex amplitudes of the magnetic and electric field are related by

$$\tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0$$

Hence show that in a good conductor the magnetic field lags the electric field by $\pi/4$ and find the ratio of their amplitudes.

In a poor conductor show that there is no phase lag and find the ratio of magnetic and electric field amplitudes. [5]

(v) Deduce the velocity of the electromagnetic wave in the limiting cases of a good conductor and in a poor conductor. [3]

10.4 **Reflection at good conductor [S]**

The half-space $z < 0$ is a vacuum, while the half space $z > 0$ is occupied by a medium with a *good* conductivity σ. A wave propagates in the z-direction ($z < 0$) with fields

$$\underline{E} = E_0\left(0, e^{i\omega((z/v)-t)}, 0\right), \qquad \underline{B} = -v^{-1}E_0\left(e^{i\omega((z/v)-t)}, 0, 0\right),$$

and is incident upon the plane $z > 0$ from the half space $z < 0$. Show that the magnitude of the reflected wave is approximately $1 - (\omega\varepsilon_0/2\sigma)^{\frac{1}{2}}$, and deduce the radiation pressure on the surface.

10.5 On reflection... [S]

In this question you may us without proof the results for the amplitude reflection and transmission coefficients

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad ; \quad t = \frac{2Z_2}{Z_2 + Z_1}$$

for a wave in medium 1 arriving at normal incidence to a planar interface with medium 2. You may also use without proof the expression for Poynting's vector (time averaged) in an EM wave propagating along $\hat{\underline{k}}$, of electric amplitude E_0, in a general medium with intrinsic impedance $Z = (\mu_r\mu_0/\varepsilon_r\varepsilon_0)^{1/2}$:

$$\langle \underline{S} \rangle = \hat{\underline{k}}\Re\left(\frac{1}{2Z}\right)|E_0|^2$$

(i) Use the definition of the intensity (magnitude of Poynting vector) and $\langle \underline{S} \rangle$ to show that the intensity reflection and transmission coefficients are given by

$$R \equiv \frac{I_{ref}}{I_{in}} = |r|^2 \qquad T \equiv \frac{I_{trans}}{I_{in}} = \frac{\Re(1/Z_2)}{\Re(1/Z_1)}|t|^2$$

Show that $R + T = 1$ in the cases: Z_1 and Z_2 are both real; Z_1 real and Z_2 complex $Z_2 = u_2 + iv_2$ (v is not velocity here)

(iii) Consider EM radiation that is normally incident on an interface between two media. Assume that ε_r and μ_r are real for both media, but **not** that $\mu_r = 1$. Show that the Poynting vectors of incident, reflected and transmitted waves obey

$$\langle \underline{S}_I \rangle - \langle \underline{S}_R \rangle = \langle \underline{S}_T \rangle$$

(iv) Using conservation arguments, explain why the corresponding result for general angle of incidence must be

$$(\langle \underline{S}_I \rangle - \langle \underline{S}_R \rangle)\cos\theta_I = \langle \underline{S}_T \rangle \cos\theta_T$$

10.6 Impeding reflection [S]

(i) Derive the amplitude transmission and reflection coefficients t and r for EM waves in a medium of intrinsic impedance Z_1, normally incident at the interface with a medium of intrinsic impedance Z_2.

(ii) A material X is nonmagnetic ($\mu_r = 1$) but has relative permittivity $\varepsilon_r = 3/2$. Another material Y is both paramagnetic and dielectric; it has $\mu_r = 6/5$ and $\varepsilon_r = 9/5$. Find the speed of of light $\tilde{c}$ in each medium.

(iii) Show that the refractive indices n_X and n_Y are not equal, but that despite this, an EM wave travelling through medium X, normally incident on a slab of medium Y, is perfectly transmitted with no reflected wave.

10.7 Seeing it from Brewster's angle [S]

The Fresnel formula for the amplitude reflection coefficient of light, polarised with $\underline{E}$ in the plane of incidence, is

$$r = \frac{\sin 2\theta_T - \sin 2\theta_I}{\sin 2\theta_T + \sin 2\theta_I}.$$

(i) To what class of media does this result apply?

(ii) Define the Brewster angle, and show it obeys $\tan\theta_B = n_2/n_1$ where n_1, n_2 are the refractive indices of the media through which light is incident and transmitted.

(iii) EM waves, polarised with E in the plane of incidence, are incident from a vacuum onto flint glass ($n = 1.73$). Find the Brewster angle, and sketch for such waves the intensity reflection coefficient $\mathcal{R}$ as a function of incident angle θ_I.

10.8 Plasma physics (in a nutshell) [C]

A plasma is an ionized gas in which few collisions occur. This means that in a field, charges accelerate: the current obeys $\partial\underline{J}/\partial t = \alpha\varepsilon_0\underline{E}$, with α a constant.

(i) Assuming that ε_r and μ_r are unity in the absence of such currents, take the curl of Faraday's law and show that the resulting wave equation is equivalent to that of a medium with $\varepsilon_r(\omega) = 1 - \alpha/\omega^2$.

(ii) Show that the dispersion relation for such a plasma is $\omega^2 = \omega_0^2 + c^2k^2$ and find ω_0 in terms of α. Deduce that no propagation is possible for frequencies below ω_0 (called the plasma frequency).

(iii) A distant astronomical object is of a class known to emit radio waves with a flat spectrum (roughly equal intensities at all observable frequencies). Between this object and the earth there is an interstellar plasma cloud with a certain ω_0. What radio spectrum would be observed on earth?