The Fourier Transform
(What you need to know)

Mathematical Background for:

Senior Honours	**Modern Optics**
Senior Honours	**Digital Image Analysis**
Senior Honours	**Optical Laboratory Projects**
MSc	**Theory of Image Processing**

Session: 2007-2008
Version: 3.1.1

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1 Introduction

Fourier Transform theory is essential to many areas of physics including acoustics and signal processing, optics and image processing, solid state physics, scattering theory, and the more generally, in the solution of differential equations in applications as diverse as weather modeling to quantum field calculations. The *Fourier Transform* can either be considered as expansion in terms of an orthogonal bases set (sine and cosine), *or* a shift of space from *real space* to *reciprocal space*. Actually these two concepts are mathematically identical although they are often used in very different physical situations.

The aim of this booklet is to cover the Fourier Theory required primarily for the

- Junior Honours course OPTICS.
- Senior Honours course MODERN OPTICS[1] and DIGITAL IMAGE ANALYSIS
- Geoscience MSc course THEORY OF IMAGE PROCESSING.

It also contains examples from acoustics and solid state physics so should be generally useful for these courses. The mathematical results presented in this booklet will be used in the above courses and they are *expected* to be known.

There are a selection of tutorial style questions with full solutions at the back of the booklet. These contain a range of examples and mathematical proofs, some of which are fairly difficult, particularly the parts in *italic*. The mathematical proofs are *not* in themselves an examinal part of the lecture courses, but the results and techniques employed are.

Further details of Fourier Transforms can be found in "*Introduction to the Fourier Transform and its Applications*" by Bracewell and "*Mathematical Methods for Physics and Engineering*" by Riley, Hobson & Bence.

1.1 Notation

Unlike many mathematical field of science, Fourier Transform theory does not have a well defined set of standard notations. The notation maintained throughout will be:

$$\begin{aligned} x,y &\rightarrow \text{Real Space co-ordinates} \\ u,v &\rightarrow \text{Frequency Space co-ordinates} \end{aligned}$$

and lower case functions (*eg* $f(x)$), being a real space function and upper case functions (*eg* $F(u)$), being the corresponding Fourier transform, thus:

$$\begin{aligned} F(u) &= \mathcal{F}\{f(x)\} \\ f(x) &= \mathcal{F}^{-1}\{F(u)\} \end{aligned}$$

where $\mathcal{F}\{\}$ is the Fourier Transform operator.

The character $\imath$ will be used to denote $\sqrt{-1}$, it should be noted that this character differs from the conventional i (or j). This slightly odd convention and is to avoid confusion when the digital version of the Fourier Transform is discussed in some courses since then i and j will be used as summation variables.

[1] not offered in 2006/2007 session.

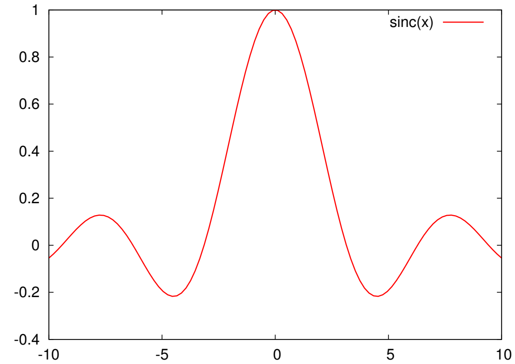


Figure 1: The sinc() function.

Two special functions will also be employed, these being sinc() defined[2] as,

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} \tag{1}$$

giving $\operatorname{sinc}(0) = 1$[3] and $\operatorname{sinc}(x_0) = 0$ at $x_0 = \pm\pi, \pm 2\pi, \ldots$, as shown in figure 1. The *top hat* function $\Pi(x)$, is given by,

$$\begin{aligned} \Pi(x) &= 1 \quad \text{for } |x| \le 1/2 \\ &= 0 \quad \text{else} \end{aligned} \tag{2}$$

being a function of unit height and width centered about $x = 0$, and is shown in figure 2

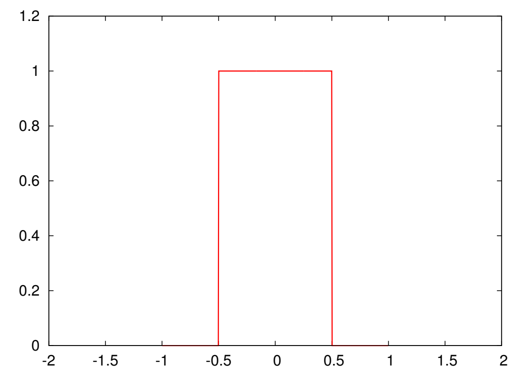


Figure 2: The $\Pi(x)$ function

2 The Fourier Transform

The definition of a one dimensional continuous function, denoted by $f(x)$, the Fourier transform is defined by:

$$F(u) = \int_{-\infty}^{\infty} f(x) \exp(-\imath 2\pi u x)\, \mathrm{d}x \tag{3}$$

[2]The sinc() function is sometimes defined with a "stray" 2π, this has the same shape and mathematical properties.

[3]See question 1

with the inverse Fourier transform defined by;

$$f(x) = \int_{-\infty}^{\infty} F(u) \exp(\imath 2\pi u x) \, \mathrm{d}u \tag{4}$$

where it should be noted that the factors of 2π are incorporated into the transform kernel[4].

Some insight to the Fourier transform can be gained by considering the case of the Fourier transform of a *real* signal $f(x)$. In this case the Fourier transform can be separated to give,

$$F(u) = F_r(u) + \imath F_\imath(u) \tag{5}$$

where we have,

$$\begin{aligned} F_r(u) &= \int_{-\infty}^{\infty} f(x) \cos(2\pi u x) \, \mathrm{d}x \\ F_\imath(u) &= -\int_{-\infty}^{\infty} f(x) \sin(2\pi u x) \, \mathrm{d}x \end{aligned}$$

So the real part of the Fourier transform is the decomposition of $f(x)$ in terms of cosine functions, and the imaginary part a decomposition in terms of sine functions. The u variable in the Fourier transform is interpreted as a frequency, for example if $f(x)$ is a sound signal with x measured in seconds then $F(u)$ is its frequency spectrum with u measured in Hertz (s^{-1}).

NOTE: Clearly (ux) *must* be dimensionless, so if x has dimensions of *time* then u *must* have dimensions of $time^{-1}$.

This is one of the most common applications for Fourier Transforms where $f(x)$ is a detected signal (for example a sound made by a musical instrument), and the Fourier Transform is used to give the spectral response.

2.1 Properties of the Fourier Transform

The Fourier transform has a range of useful properties, some of which are listed below. In most cases the proof of these properties is simple and can be formulated by use of equation 3 and equation 4.. The proofs of many of these properties are given in the questions and solutions at the back of this booklet.

Linearity: The Fourier transform is a linear operation so that the Fourier transform of the sum of two functions is given by the sum of the individual Fourier transforms. Therefore,

$$\mathcal{F}\{a f(x) + b g(x)\} = a F(u) + b G(u) \tag{6}$$

where $F(u)$ and $G(u)$ are the Fourier transforms of $f(x)$ and and $g(x)$ and a and b are constants. This property is central to the use of Fourier transforms when describing *linear* systems.

Complex Conjugate: The Fourier transform of the *Complex Conjugate* of a function is given by

$$\mathcal{F}\{f^*(x)\} = F^*(-u) \tag{7}$$

[4]There are various definitions of the Fourier transform that puts the 2π either inside the kernel or as external scaling factors. The difference between them whether the variable in Fourier space is a "frequency" or "angular frequency". The difference between the definitions are clearly just a scaling factor. The optics and digital Fourier applications the 2π is usually defined to be inside the kernel but in solid state physics and differential equation solution the 2π constant is usually an external scaling factor.

where $F(u)$ is the Fourier transform of $f(x)$.

Forward and Inverse: We have that

$$\mathcal{F}\{F(u)\} = f(-x) \tag{8}$$

so that if we apply the Fourier transform twice to a function, we get a spatially reversed version of the function. Similarly with the inverse Fourier transform we have that,

$$\mathcal{F}^{-1}\{f(x)\} = F(-u) \tag{9}$$

so that the Fourier and inverse Fourier transforms differ only by a sign.

Differentials: The Fourier transform of the derivative of a functions is given by

$$\mathcal{F}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = \imath 2\pi u F(u) \tag{10}$$

and the second derivative is given by

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -(2\pi u)^2 F(u) \tag{11}$$

This property will be used in the DIGITAL IMAGE ANALYSIS and THEORY OF IMAGE PROCESSING course to form the derivative of an image.

Power Spectrum: The *Power Spectrum* of a signal is defined by the modulus square of the Fourier transform, being $|F(u)|^2$. This can be interpreted as the *power* of the frequency components. Any function and its Fourier transform obey the condition that

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \,\mathrm{d}u \tag{12}$$

which is frequently known as *Parseval's Theorem*[5]. If $f(x)$ is interpreted at a voltage, then this theorem states that the *power* is the same whether measured in real (time), or Fourier (frequency) space.

2.2 Two Dimensional Fourier Transform

Since the three courses covered by this booklet use two-dimensional scalar potentials or images we will be dealing with two dimensional function. We will define the two dimensional Fourier transform of a continuous function $f(x,y)$ by,

$$F(u,v) = \iint f(x,y) \exp(-\imath 2\pi(ux+vy)) \,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

with the inverse Fourier transform defined by;

$$f(x,y) = \iint F(u,v) \exp(\imath 2\pi(ux+vy)) \,\mathrm{d}u\,\mathrm{d}v \tag{14}$$

where the limits of integration are taken from $-\infty \rightarrow \infty$[6]

[5]Strictly speaking Parseval's Theorem applies to the case of Fourier series, and the equivalent theorem for Fourier transforms is correctly, but less commonly, known as Rayleigh's theorem

[6]Unless otherwise specified all integral limits will be assumed to be from $-\infty \rightarrow \infty$

Again for a real two dimensional function $f(x,y)$, the Fourier transform can be considered as the decomposition of a function into its sinusoidal components. If $f(x,y)$ is considered to be an image with the "brightness" of the image at point (x_0,y_0) given by $f(x_0,y_0)$, then variables x,y have the dimensions of length. In Fourier space the variables u,v have therefore the dimensions of *inverse* length, which is interpreted as *Spatial Frequency*.

NOTE: Typically x and y are measured in mm so that u and v have are in units of mm^{-1} also referred to at *lines per mm*.

The Fourier transform can then be taken as being the decomposition of the image into two dimensional sinusoidal *spatial* frequency components. This property will be examined in greater detail the relevant courses.

The properties of one the dimensional Fourier transforms covered in the previous section convert into two dimensions. Clearly the derivatives then become

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = \imath 2\pi u F(u,v) \tag{15}$$

and with

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = \imath 2\pi v F(u,v) \tag{16}$$

yielding the important result that,

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v) \tag{17}$$

where we have that $w^2 = u^2 + v^2$. So that taking the Laplacian of a function in real space is equivalent to multiplying its Fourier transform by a circularly symmetric quadratic of $-4\pi^2 w^2$.

The two dimensional Fourier Transform $F(u,v)$, of a function $f(x,y)$ is a separable operation, and can be written as,

$$F(u,v) = \int P(u,y)\exp(-\imath 2\pi v y)\,\mathrm{d}y \tag{18}$$

where

$$P(u,y) = \int f(x,y)\exp(-\imath 2\pi u x)\,\mathrm{d}x \tag{19}$$

where $P(u,y)$ is the Fourier Transform of $f(x,y)$ with respect to x only. This property of separability will be considered in greater depth with regards to digital images and will lead to an implementation of two dimensional discrete Fourier Transforms in terms of one dimensional Fourier Transforms.

2.3 The Three-Dimensional Fourier Transform

In the three dimensional case we have a function $f(\vec{r})$ where $\vec{r} = (x,y,z)$, then the three-dimensional Fourier Transform

$$F(\vec{s}) = \iiint f(\vec{r})\exp(-\imath 2\pi \vec{r}.\vec{s})\,\mathrm{d}\vec{r}$$

where $\vec{s} = (u,v,w)$ being the three reciprocal variables each with units length^-1. Similarly the inverse Fourier Transform is given by

$$f(\vec{r}) = \iiint F(\vec{s})\exp(\imath 2\pi \vec{r}.\vec{s})\,\mathrm{d}\vec{s}$$

This is used extensively in solid state physics where the three-dimensional Fourier Transform of a crystal structures is usually called *Reciprocal Space*[7].

The three-dimensional Fourier Transform is again separable into one-dimensional Fourier Transform. This property is independent of the dimensionality and multi-dimensional Fourier Transform can be formulated as a series of one dimensional Fourier Transforms.

3 Dirac Delta Function

A frequently used concept in Fourier theory is that of the *Dirac Delta Function*, which is somewhat abstractly defined as:

$$\begin{aligned} \delta(x) &= 0 \qquad \text{for } x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x &= 1 \end{aligned} \tag{20}$$

This can be thought of as a very "*tall*-and-*thin*" spike with unit area located at the origin, as shown in figure 3.

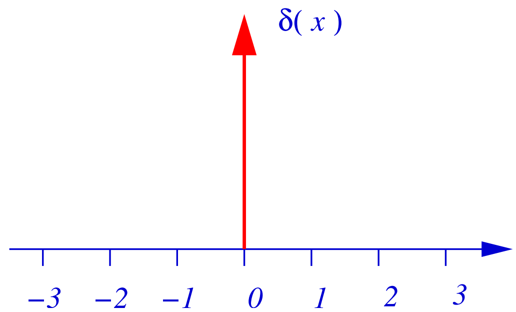


Figure 3: The δ-function.

NOTE: The δ-functions should **not** be considered to be an *infinitely high* spike of *zero* width since it scales as:

$$\int_{-\infty}^{\infty} a\,\delta(x)\,\mathrm{d}x = a$$

where a is a constant.

The *Delta Function* is not a true function in the analysis sense and if often called an *improper function*. There are a range of definitions of the *Delta Function* in terms of *proper function*, some of which are:

$$\begin{aligned} \Delta_\varepsilon(x) &= \frac{1}{\varepsilon\sqrt{\pi}} \exp\left(\frac{-x^2}{\varepsilon^2}\right) \\ \Delta_\varepsilon(x) &= \frac{1}{\varepsilon}\Pi\left(\frac{x-\frac{1}{2}\varepsilon}{\varepsilon}\right) \\ \Delta_\varepsilon(x) &= \frac{1}{\varepsilon}\mathrm{sinc}\left(\frac{x}{\varepsilon}\right) \end{aligned}$$

[7] This is also referred to as $\vec{k}$-space where $\vec{k} = 2\pi\vec{s}$

being the Gaussian, Top-Hat and Sinc approximations respectively. All of these expressions have the property that,

$$\int_{-\infty}^{\infty} \Delta_\varepsilon(x)\,\mathrm{d}x = 1 \quad \forall\varepsilon \tag{21}$$

and we may form the approximation that,

$$\delta(x) = \lim_{\varepsilon\to 0} \Delta_\varepsilon(x) \tag{22}$$

which can be interpreted as making any of the above approximations $\Delta_\varepsilon(x)$ a very "*tall*-and*thin*" spike with unit area.

In the field of optics and imaging, we are dealing with two dimensional distributions, so it is especially useful to define the *Two Dimensional Dirac Delta Function*, as,

$$\begin{aligned} \delta(x,y) &= 0 \qquad \text{for } x \neq 0\ \&\ y \neq 0 \\ \iint \delta(x,y)\,\mathrm{d}x\,\mathrm{d}y &= 1 \end{aligned} \tag{23}$$

which is the two dimensional version of the $\delta(x)$ function defined above, and in particular:

$$\delta(x,y) = \delta(x)\,\delta(y). \tag{24}$$

This is the two dimensional analogue of the *impulse* function used in signal processing. In terms of an imaging system, this function can be considered as a single bright spot in the centre of the field of view, for example a single bright star viewed by a telescope.

3.1 Properties of the Dirac Delta Function

Since the *Dirac Delta Function* is used extensively, and has some useful, and slightly perculiar properties, it is worth considering these are this point. For a function $f(x)$, being integrable, then we have that

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\,\mathrm{d}x = f(0) \tag{25}$$

which is often taken as an alternative definition of the Delta function. This says that integral of any function multiplied by a δ-function located about zero is just the value of the function at zero. This concept can be extended to give the *Shifting Property*, again for a function $f(x)$, giving,

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\,\mathrm{d}x = f(a) \tag{26}$$

where $\delta(x-a)$ is just a δ-function located at $x = a$ as shown in figure 4.

In two dimensions, for a function $f(x,y)$, we have that,

$$\iint \delta(x-a, y-b)\, f(x,y)\,\mathrm{d}x\,\mathrm{d}y = f(a,b) \tag{27}$$

where $\delta(x-a, y-b)$ is a δ-function located at position a,b. This property is central to the idea of convolution, which is used extensively in image formation theory, and in digital image processing.

The Fourier transform of a Delta function is can be formed by direct integration of the definition of the Fourier transform, and the shift property in equation 25 above. We get that,

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\, \exp(-\imath 2\pi u x)\,\mathrm{d}x = \exp(0) = 1 \tag{28}$$

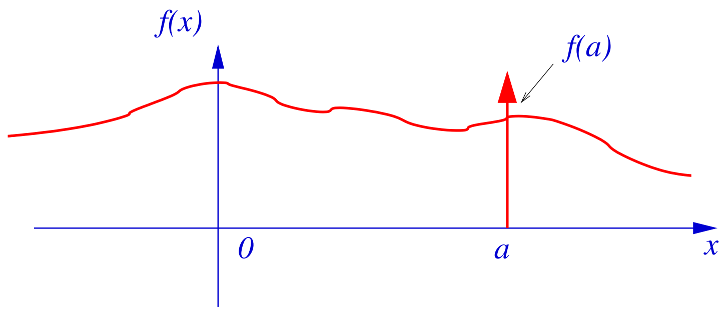


Figure 4: Shifting property of the δ-function.

and then by the *Shifting Theorem*, equation 26, we get that,

$$\mathcal{F}\{\delta(x-a)\} = \exp(-\imath 2\pi a u) \tag{29}$$

so that the Fourier transform of a shifted Delta Function is given by a phase ramp. It should be noted that the modulus squared of equation 29 is

$$|\mathcal{F}\{\delta(x-a)\}|^2 = |\exp(-\imath 2\pi a u)|^2 = 1$$

saying that the power spectrum a *Delta Function* is a constant independent of its location in real space.

Now noting that the Fourier transform is a linear operation, then if we consider two *Delta Function* located at $\pm a$, then from equation 29 the Fourier transform gives,

$$\mathcal{F}\{\delta(x-a)+\delta(x+a)\} = \exp(-\imath 2\pi a u) + \exp(\imath 2\pi a u) = 2\cos(2\pi a u) \tag{30}$$

while if we have the *Delta Function* at $x = -a$ as negative, then we also have that,

$$\mathcal{F}\{\delta(x-a)-\delta(x+a)\} = \exp(-\imath 2\pi a u) - \exp(\imath 2\pi a u) = -2\imath\sin(2\pi a u). \tag{31}$$

Noting the relations between forward and inverse Fourier transform we then get the two useful results that

$$\mathcal{F}\{\cos(2\pi a x)\} = \frac{1}{2}[\delta(u-a)+\delta(u+a)] \tag{32}$$

and that

$$\mathcal{F}\{\sin(2\pi a x)\} = \frac{1}{2\imath}[\delta(u-a)-\delta(u+a)] \tag{33}$$

So that the Fourier transform of a cosine or sine function consists of a single frequency given by the period of the cosine or sine function as would be expected.

3.2 The Infinite Comb

If we have an infinite series of Delta functions at a regular spacing of Δx, this is described as an *Infinite Comb*. The the expression for a *Comb* is given by,

$$\mathrm{Comb}_{\Delta x}(x) = \sum_{i=-\infty}^{\infty} \delta(x - i\Delta x). \tag{34}$$

A short section of such a *Comb* is shown in figure 5.

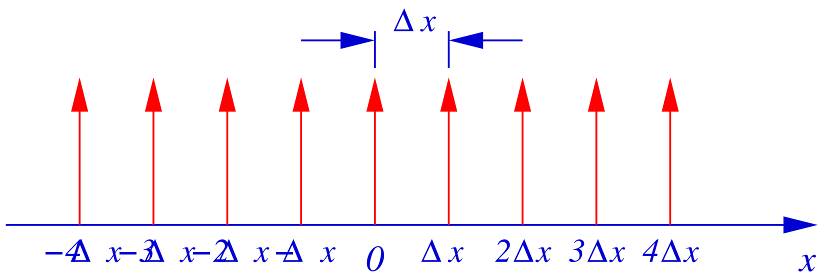


Figure 5: Infinite Comb with separation Δx

Since the Fourier transform is a linear operation then the Fourier transform of the infinite comb is the sum of the Fourier transforms of shifted Delta functions, which from equation (29) gives,

$$\mathcal{F}\left\{\mathrm{Comb}_{\Delta x}(x)\right\} = \sum_{i=-\infty}^{\infty} \exp(-\imath 2\pi i \Delta x u) \tag{35}$$

Now the exponential term,

$$\exp(-\imath 2\pi i \Delta x u) = 1 \quad \text{when } 2\pi \Delta x u = 2\pi n$$

so that:

$$\begin{aligned}\sum_{i=-\infty}^{\infty} \exp(-\imath 2\pi i \Delta x u) &\rightarrow \infty \quad \text{when } u = \tfrac{n}{\Delta x} \\ &= 0 \quad \text{else}\end{aligned}$$

which is an infinite series of δ-function at a separation of $\Delta u = \frac{1}{\Delta x}$. So that an *Infinite Comb* Fourier transforms to another *Infinite Comb* or reciprocal spacing,

$$\mathcal{F}\left\{\mathrm{Comb}_{\Delta x}(x)\right\} = \mathrm{Comb}_{\Delta u}(u) \quad \text{with } \Delta u = \tfrac{1}{\Delta x} \tag{36}$$

This is an important result used in Sampling Theory in the DIGITAL IMAGE ANALYSIS and IMAGE PROCESSING I courses.

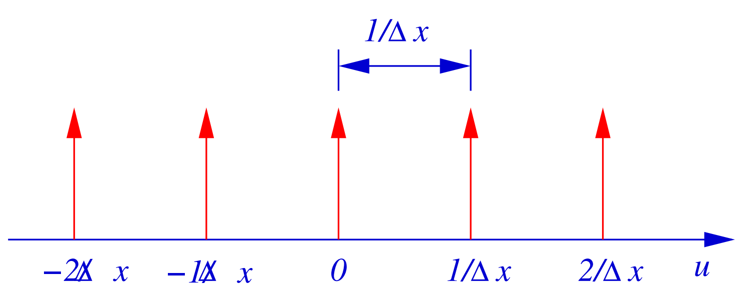


Figure 6: Fourier Transform of comb function.

4 Symmetry Conditions

When we take the the Fourier Transform of a *real* function, for example a one-dimensional sound signal or a two-dimensional image we obtain a *complex* Fourier Transform. This Fourier

Transform has special symmetry properties that are essential when calculating and/or manipulating Fourier Transforms. This section it of the booklet is mainly aimed at the DIGITAL IMAGE ANALYSIS and THEORY OF IMAGE PROCESSING courses that make extensive use of these symmetry conditions.

4.1 One-Dimensional Symmetry

Firstly consider the case of a one dimensional real function $f(x)$, with a Fourier transform of $F(u)$. Since $f(x)$ is real then from previous we can write

$$F(u) = F_r(u) + \imath F_\imath(u)$$

where the real and imaginary parts are given by the cosine and sine transforms to be

$$\begin{aligned} F_r(u) &= \int f(x)\cos(2\pi ux)\,\mathrm{d}x \\ F_\imath(u) &= -\int f(x)\sin(2\pi ux)\,\mathrm{d}x \end{aligned} \tag{37}$$

now cos() is a symmetric function and sin() is an anti-symmetric function, as shown in figure 7, so that:

$$\begin{array}{ll} F_r(u) & \text{is Symmetric} \\ F_\imath(u) & \text{is Anti-symmetric} \end{array}$$

which can be written out explicitly as,

$$\begin{aligned} F_r(u) &= F_r(-u) \\ F_\imath(u) &= -F_\imath(-u) \end{aligned} \tag{38}$$

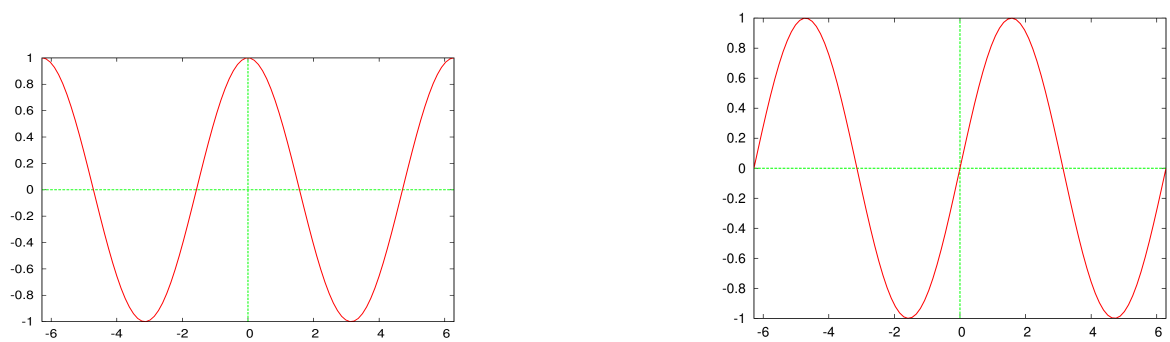


Figure 7: Symmetry properties of cos() and sin() functions

The *power spectrum* is given by

$$|F(u)|^2 = F_r(u)^2 + F_\imath(u)^2$$

so that if the real and imaginary parts obey the symmetry property given in equation (38), then clearly the *power spectrum* is also symmetric with

$$|F(u)|^2 = |F(-u)|^2 \tag{39}$$

so when the power spectrum of a signal is calculated it is normal to display the signal from $0 \rightarrow u_{\max}$ and ignore the negative components.

4.2 Two-Dimensional Symmetry

In two dimensional we have a real image $f(x,y)$, and then as above the Fourier transform of this image can be written as,

$$F(u,v) = F_r(u,v) + \imath F_\imath(u,v) \tag{40}$$

where after expansion of the exp() functions into cos() and sin() functions we get that

$$F_r(u,v) = \iint f(x,y)\,[\cos(2\pi ux)\,\cos(2\pi vy) - \sin(2\pi ux)\,\sin(2\pi vy)]\,\mathrm{d}x\,\mathrm{d}y$$

and that;

$$F_\imath(u,v) = \iint f(x,y)\,[\cos(2\pi ux)\,\sin(2\pi vy) + \sin(2\pi ux)\,\cos(2\pi vy)]\,\mathrm{d}x\,\mathrm{d}y$$

In this case the symmetry properties are more complicated, however we say that the real part is symmetric and the imaginary part is anti-symmetric, where in two dimensions the symmetry conditions are given by,

$$\begin{aligned} F_r(u,v) &= F_r(-u,-v) \\ F_r(-u,v) &= F_r(u,-v) \end{aligned} \tag{41}$$

for the real part of the Fourier transform, and

$$\begin{aligned} F_\imath(u,v) &= -F_\imath(-u,-v) \\ F_\imath(-u,v) &= -F_\imath(u,-v) \end{aligned} \tag{42}$$

for the imaginary part. Similarly the two dimensional *power spectrum* is also symmetric, with

$$\begin{aligned} |F(u,v)|^2 &= |F(-u,-v)|^2 \\ |F(-u,v)|^2 &= |F(u,-v)|^2 \end{aligned} \tag{43}$$

This symmetry condition is shown schematically in figure 8, which shows a series of symmetric points.

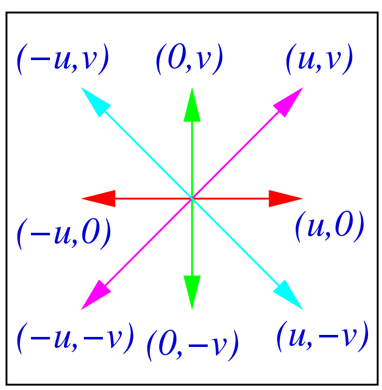


Figure 8: Symmetry in two dimensions

These symmetry properties has a major significance in the digital calculation of Fourier transforms and the design of digital filters, which is discussed in greater detail in the relevant courses.

5 Convolution of Two Functions

The concept of *convolution* is central to Fourier theory and the analysis of Linear Systems. In fact the convolution property is what really makes Fourier methods useful. In one dimension the convolution between two functions, $f(x)$ and $h(x)$ is defined as:

$$g(x) = f(x) \odot h(x) = \int_{-\infty}^{\infty} f(s)\, h(x-s)\, \mathrm{d}s \tag{44}$$

where s is a dummy variable of integration. This operation may be considered the *area of overlap* between the function $f(x)$ and the *spatially reversed version* of the function $h(x)$. The result of the convolution of two simple one dimensional functions is shown in figure 9.

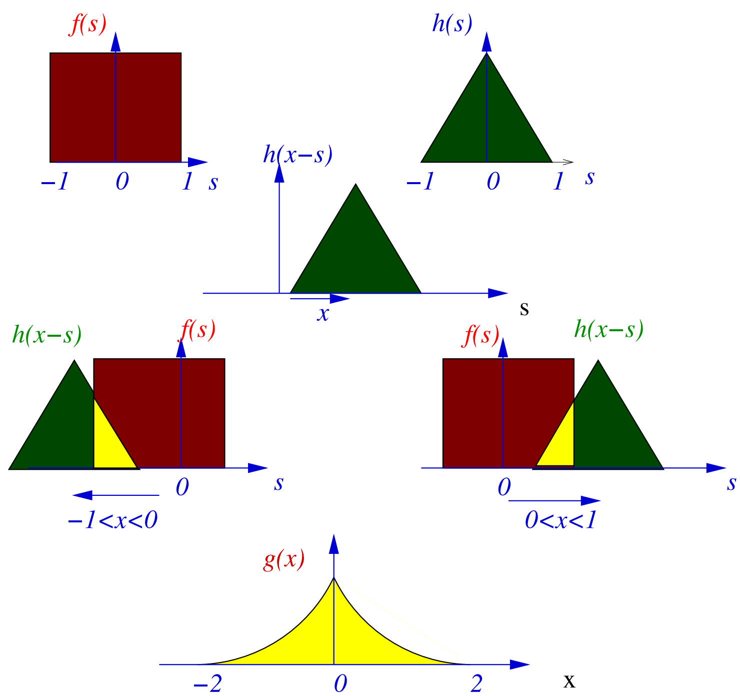


Figure 9: Convolution of two simple functions.

The *Convolution Theorem* relates the convolution between the real space domain to a multiplication in the Fourier domain, and can be written as;

$$G(u) = F(u)\, H(u) \tag{45}$$

where

$$\begin{aligned} G(u) &= \mathcal{F}\{g(x)\} \\ F(u) &= \mathcal{F}\{f(x)\} \\ H(u) &= \mathcal{F}\{h(x)\} \end{aligned}$$

This is the most important result in this booklet and will be used extensively in all three courses. This concept may appear a bit abstract at the moment but there will be extensive illustrations of convolution throughout the courses.

5.1 Simple Properties

The convolution is a linear operation which is distributative, so that for three functions $f(x)$, $g(x)$ and $h(x)$ we have that

$$f(x) \odot (g(x) \odot h(x)) = (f(x) \odot g(x)) \odot h(x) \tag{46}$$

and commutative, so that

$$f(x) \odot h(x) = h(x) \odot f(x) \tag{47}$$

If the two functions $f(x)$ and $h(x)$ are of finite extent, (are zero outwith a finite range of x), then the extent (or *width*) of the convolution $g(x)$ is given by the sum of the widths the two functions. For example if figure 9 both $f(x)$ and $h(x)$ non-zero over the finite range $x = \pm 1$ which the convolution $g(x)$ is non-zero over the range $x = \pm 2$. This property will be used in optical image formation and in the practical implication of convolution filters in digital image processing.

The special case of the convolution of a function with a $\text{Comb}(x)$ function results in replication of the function at the comb spacing as shown in figure 10. Clearly if the extent of the function is less than the comb spacing, as shown in this figure, the replications are separated, while if the the extent of the function is greater than the comb period, overlap of adjacent replications will occur. This operation is central to sampling theory, and image formation and will be discussed in details in the relevant courses. This idea is also central to Solid State Physics where the electron density of a unit cell is convolved with the lattice sites.

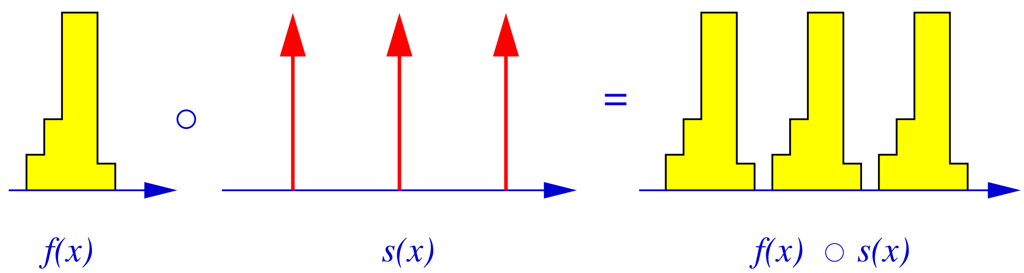


Figure 10: Convolution of function with comb of δ-functions.

5.2 Two Dimensional Convolution

As with Fourier Transform the extension to two-dimensions is simple with,

$$g(x,y) = f(x,y) \odot h(x,y) = \iint f(s,t)\, h(x-s, y-t)\, \mathrm{d}s\, \mathrm{d}t \tag{48}$$

which in the Fourier domain gives the important result that,

$$G(u,v) = F(u,v)\, H(u,v) \tag{49}$$

This relation is fundamental to both optics and image processing and will be used extensively in the both courses.

The most important implication of the *Convolution Theorem* is that,

Multiplication in Real Space $\Longleftrightarrow$ Convolution in Fourier Space
Convolution in Real Space $\Longleftrightarrow$ Multiplication in Fourier Space

which is a **Key Result**.

6 Correlation of Two Functions

A closely related operation to *Convolution* is the operation of *Correlation* of two functions. In *Correlation* two function are shifted and the area of overlap formed by integration, but this time *without* the spatial reversal involved in *convolution*. The *Correlation* between two function $f(x)$ and $h(x)$ is given by

$$c(x) = f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(s)\, h^*(s-x)\, \mathrm{d}s \tag{50}$$

where $h^*(x)$ is the *complex conjugate* of $h(x)$[8]. This operation is shown for two simple functions in figure 11. Comparison between the *convolution* in figure 9 and the *correlation* shown that the only difference is that the second function is *not* spatially reversed and the direction of the shift is changed.

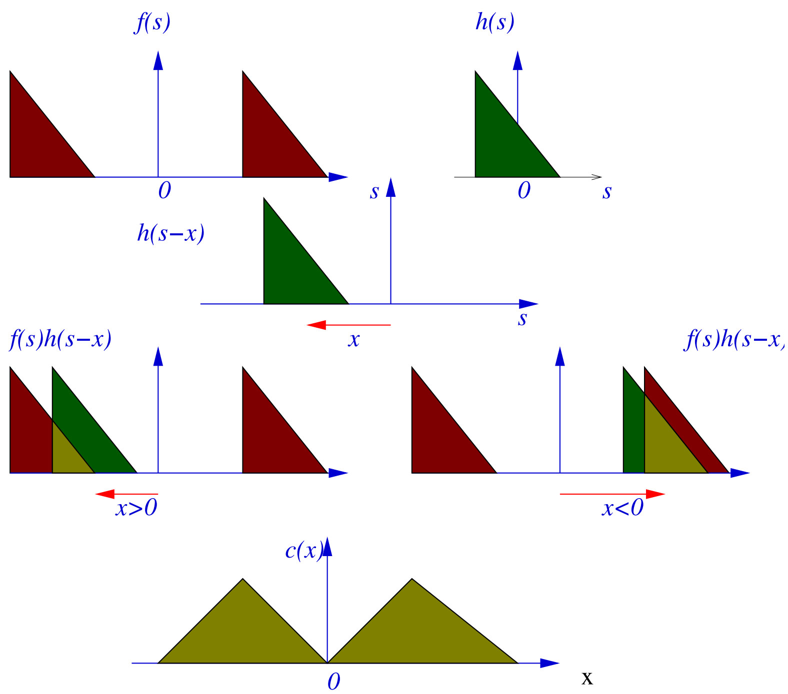


Figure 11: Correlation of two simple functions.

Of more importance, if we consider $f(x)$ to be the “signal” and $h(x)$ to be the “target” then we see that the correlation gives a peak where the “signal” matches the “target”. This gives the basis of the simples method of target detection[9].

In the Fourier Domain the *Correlation Theorem* becomes

$$C(u) = F(u)\, H^*(u) \tag{51}$$

where

$$C(u) \quad = \quad \mathcal{F}\{c(x)\}$$

[8] It should be noted that for a real function *complex conjugation* does not effect the function, so if both $f(x)$ and $h(x)$ are *real* then the *Convolution* and *Correlation* differ only by a change of sign, which represents the spatial reversal on one of the functions.

[9] The two-dimensional version of this is considered in question 9.

$$\begin{aligned} F(u) &= \mathcal{F}\{f(x)\} \\ H(u) &= \mathcal{F}\{h(x)\} \end{aligned}$$

It should be noted that the Fourier Transform $H(u)$ is generally complex, and the *complex conjugation is* of vital significance to the operation.

This is again a linear operation, which is distributative, but however is **not** commutative, since if

$$c(x) = f(x) \otimes h(x)$$

then we can show that

$$h(x) \otimes f(x) = c^*(-x)$$

In two dimensions we have the correlation between two functions given by

$$c(x,y) = f(x,y) \otimes h(x,y) = \iint f(s,t)\, h^*(s-x, t-y)\, \mathrm{d}s\, \mathrm{d}t \tag{52}$$

which in Fourier space gives,

$$C(u,v) = F(u,v)\, H^*(u,v) \tag{53}$$

Correlation is used in optics to to characterise the incoherent optical properties of a system and in digital imaging as a measure of the "similarity" between two images.

6.1 Autocorrelation

If we consider the special case of *correlation* with two identical real space functions, we obtain the *correlation* of the input function with itself, being known as the *Autocorrelation*, being,

$$a(x,y) = f(x,y) \otimes f(x,y) \tag{54}$$

so that in Fourier space we have,

$$A(u,v) = F(u,v)\, F^*(u,v) = |F(u,v)|^2 \tag{55}$$

which is the *Power Spectrum* of the function $f(x,y)$. Therefore the *Autocorrelation* of a function is given by the *Inverse Fourier Transform* of the *Power Spectrum*, giving,

$$a(x,y) = \mathcal{F}^{-1}\left\{|F(u,v)|^2\right\} \tag{56}$$

In this case the *correlation* must be commutative, so we have that

$$a^*(-x,-y) = a(x,y)$$

If in addition the function $f(x)$ is real, then clearly the correlation of a real function with it self is real, so that $a(x)$ is real. Therefore for a real function the autocorrelation is symmetric.

Workshop Questions

7 Questions

7.1 The sinc() function

State the expression for $\mathrm{sinc}(x)$ in terms of $\sin(x)$, and prove that

$$\mathrm{sinc}(0) = 1$$

Sketch the graph of

$$y = \mathrm{sinc}(ax) \quad \text{and} \quad y = \mathrm{sinc}^2(ax)$$

where a is a constant, and identify the locations of the zeros in each case.

Solution

The definition of $\mathrm{sinc}(x)$ is

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

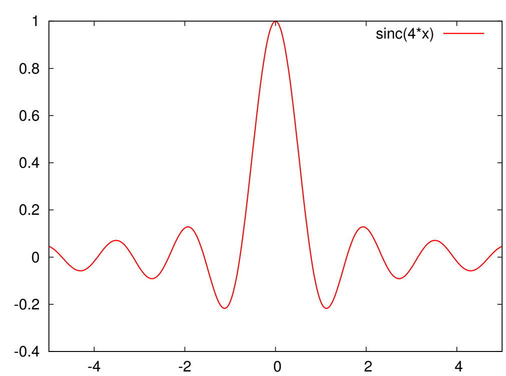
To find the value as $x \longrightarrow 0$ take the Taylor expansion about $x = 0$ to get,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

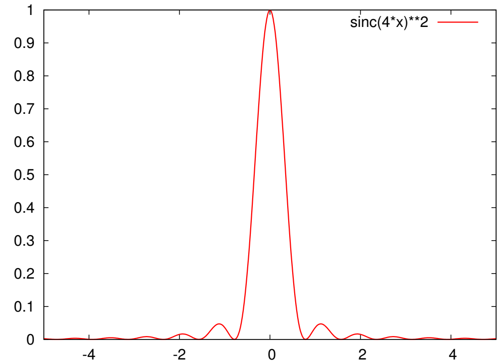
so we have that

$$\mathrm{sinc}(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

so, when $x = 0$ then $\mathrm{sinc}(0) = 1$ as expected.
Sketch of $\mathrm{sinc}(ax)$ when $a = 4$



and sketch of $\mathrm{sinc}^2(ax)$ when $a = 4$



Both functions have zero in the same place, when $ax = \pm n\pi$, so at

$$x_n = \pm \frac{n\pi}{a} \quad n = 1, 2, \ldots$$

Note the **larger** a the closer the zero are together.

7.2 Rectangular Aperture

Calculate the two dimensional Fourier transform of a rectangle of unit height and size a by b centered about the origin.

If $a = 5\,\text{mm}$ and $b = 1\,\text{mm}$ calculate the location of first zeros in the u and v direction. Sketch the real part of the Fourier transform. (`Maple` or `gnuplot` experts can make nice plots)

Solution

We can express a rectangle of size $a \times b$ by:

$$\begin{aligned} f(x,y) &= 1 \quad |x| < a/2 \text{ and } |y| < b/2 \\ &= 0 \quad \text{else} \end{aligned}$$

the Fourier Transform is given by:

$$F(u,v) = \iint f(x,y)\, \exp\left(-\imath 2\pi(ux+vy)\right) \mathrm{d}x\,\mathrm{d}y$$

which can then be written as:

$$F(u,v) = \int_{-b/2}^{b/2} \left[\int_{-a/2}^{a/2} \exp\left(-\imath 2\pi(ux+vy)\right) \mathrm{d}x \right] \mathrm{d}y$$

Noting that the exp() term is separable, this can be written as

$$F(u,v) = \int_{-b/2}^{b/2} \exp(-\imath 2\pi v y)\, \mathrm{d}y \int_{-a/2}^{a/2} \exp(-\imath 2\pi u x)\, \mathrm{d}x$$

Look at one of the integrals, and we get,

$$\begin{aligned} \int_{-a/2}^{a/2} \exp(-\imath 2\pi u x)\, \mathrm{d}x &= \frac{1}{-\imath 2\pi u} \left[\exp(-\imath 2\pi u x\right]_{-a/2}^{a/2} \\ &= \frac{-\imath}{2\pi u} \left[\exp(\imath \pi a u) - \exp(-\imath \pi u a)\right] \\ &= \frac{\sin(\pi a u)}{\pi u} \\ &= a\,\mathrm{sinc}(\pi a u) \end{aligned}$$

The other integral is of exactly the same form, so that the Fourier transform of the rectangle is:

$$F(u,v) = ab\,\mathrm{sinc}(\pi a u)\,\mathrm{sinc}(\pi b v)$$

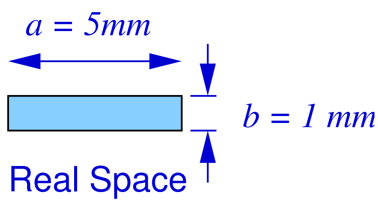
The zero of this function occur at:

$$\begin{aligned} u_n &= \pm\frac{n}{a} \quad \text{for } n = 1,2,3,\ldots \\ v_m &= \pm\frac{m}{b} \quad \text{for } m = 1,2,3,\ldots \end{aligned}$$

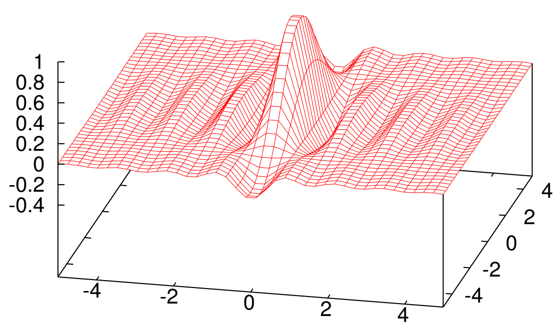
which if $a = 5\,\mathrm{mm}$ and $b = 1\,\mathrm{mm}$ then

$$\begin{aligned} u_n &= 0.2\,\mathrm{mm}^{-1}, 0.4\,\mathrm{mm}^{-1}, 0.6\,\mathrm{mm}^{-1}, \ldots \\ v_n &= 1\,\mathrm{mm}^{-1}, 2\,\mathrm{mm}^{-1}, 3\,\mathrm{mm}^{-1}, \ldots \end{aligned}$$

In diagrams we get,



so in Fourier space we get a three-Dimensional plot plot of



Note that the *long/thin* shape of the rectangle Fourier Transforms to *tall/thin* structures in the Fourier Transform.

7.3 Gaussians

Calculate the Fourier Transform of a two-dimensional Gaussian given by,

$$f(x,y) = \exp\left(-\frac{r^2}{r_0^2}\right)$$

where $r^2 = x^2 + y^2$ and r_0 is the radius of the e^{-1} point.
You may use the standard mathematical identity that

$$\int_{-\infty}^{\infty} \exp(-bx^2)\ \exp(iax)\,\mathrm{d}x = \sqrt{\frac{\pi}{b}}\ \exp\left(-\frac{a^2}{4b}\right)$$

Solution

The Fourier Transform is given by:

$$F(u,v) = \iint \exp\left(-\frac{(x^2+y^2)}{r_0^2}\right) \exp\left(-\imath 2\pi(ux+vy)\right) \mathrm{d}x\mathrm{d}y$$

Since the Gaussian *and* the Fourier kernel are separable, this can be written as

$$F(u,v) = \int \exp\left(-\frac{x^2}{r_0^2}\right) \exp(-\imath 2\pi ux)\mathrm{d}x \int \exp\left(-\frac{y^2}{r_0^2}\right) \exp(-\imath 2\pi vy)\mathrm{d}y$$

so we need only evaluate one integral.
Noting the result given that

$$\int_{-\infty}^{\infty} \exp(-bx^2)\exp(iax)\mathrm{d}x = \sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

See "Mathematical Handbook", M.R. Spiegel, McGraw-Hill, Page 98, Definite Integral 15.73.
The given identity is actually,

$$\int_{0}^{\infty} \exp(-bx^2)\cos(ax)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

but this can be extended to the $\infty \to \infty$ exp() integral required by noting that the cos() is symmetric so $-\infty \to \infty$ integral is *double* the $0 \to \infty$ integral and that sin() is anti-symmetric so the imaginary part of the integral from $\infty \to \infty$ is zero.
Then if be let $b = 1/r_0^2$ and $a = 2\pi u$, then

$$\int \exp\left(-\frac{x^2}{r_0^2}\right) \exp(-\imath 2\pi ux)\mathrm{d}x = \frac{\sqrt{\pi}}{r_0}\exp\left(-\pi^2 r_0^2 u^2\right)$$

which is also a Gaussian.
Key Result: The Fourier Transform of a *Gaussian* is a *Gaussian*. It is the *only function* that is its own Fourier Transform.
Exactly the same expression for the y integral, so we get that

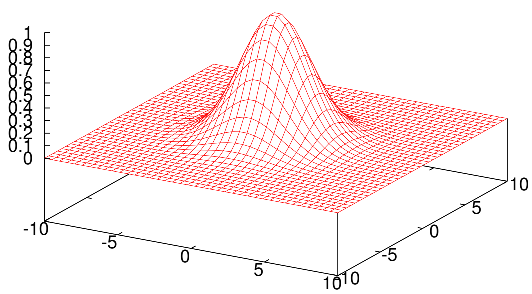
$$F(u,v) = \frac{\pi}{r_0^2}\exp\left(-\pi^2 r_0^2(u^2+v^2)\right)$$

which is more conveniently written as:

$$F(u,v) = \frac{\pi}{r_0^2}\exp\left(-\frac{w^2}{w_0^2}\right)$$

where $w^2 = u^2 + v^2$ and $w_0 = 1/\pi r_0$, which is a circular Gaussian radius with e^{-1} point at w_0. So the Fourier Transform of a Gaussian is a Gaussian of reciprocal width. Or more simply, as a *wide Gaussian* Fourier Transform for give a *narrow Gaussian* and vice versa.

General shape of two dimensional Gaussian with $r_0 = 3$ is given by



7.4 Differentials

Show, for a two dimensional function $f(x,y)$, that,

$$\mathcal{F}\left\{\frac{\partial f(x)}{\partial x}\right\} = \imath 2\pi u F(u)$$

and that

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi w)^2 F(u,v)$$

where $w^2 = u^2 + v^2$.

Solution

If $F(u)$ is the Fourier Transform of $f(x)$ then

$$f(x) = \mathcal{F}^{-1}\{F(u)\}$$

which we can write out as:

$$f(x) = \int F(u) \exp(\imath 2\pi u x)\, \mathrm{d}u$$

take differential of both sides,

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \int \imath 2\pi u F(u) \exp(\imath 2\pi u x)\, \mathrm{d}u$$

showing that the left side is

$$\mathcal{F}^{-1}\{\imath 2\pi u F(u)\}$$

take the forward Fourier transform of each side to give:

$$\mathcal{F}\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\} = \imath 2\pi u F(u)$$

as required.
In two dimensions we have the a similar result that:

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial x}\right\} = \imath 2\pi u F(u,v)$$

and that:

$$\mathcal{F}\left\{\frac{\partial f(x,y)}{\partial y}\right\} = \imath 2\pi v F(u,v)$$

the second order differentials are thus:

$$\mathcal{F}\left\{\frac{\partial^2 f(x,y)}{\partial x^2}\right\} = -(2\pi u)^2 F(u,v)$$

and that:

$$\mathcal{F}\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\} = -(2\pi v)^2 F(u,v)$$

The Laplacian,

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

so noting that the Fourier Transform is a linear relation, we get that

$$\mathcal{F}\left\{\nabla^2 f(x,y)\right\} = -(2\pi)^2(u^2+v^2)F(u,v) = -(2\pi w)^2 F(u,v)$$

as required.
The result that taking the Laplacian in real space is equivalent to multiplying by a parabolic term in Fourier space is used in image processing to detect edges.

7.5 Delta Functions

Use one of the analytic definitions of the δ-function to show that

$$\mathcal{F}\left\{\delta(x)\right\} = 1$$

Solution

Take the Top-Hat definition of the δ-function, with

$$\Delta_\varepsilon(x) = \frac{1}{\varepsilon}\Pi\left(\frac{x}{\varepsilon}\right)$$

The Gaussian definition is similar, but the sinc() *definition is a bit more difficult since it Fourier Transform to give a* $\Pi()$ *which is not actually analytic.*
The Fourier Transform is given by:

$$\mathcal{F}\left\{\Delta_\varepsilon(x)\right\} = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \exp(-\imath 2\pi u x)\,\mathrm{d}x$$

which we can integrate to give

$$\frac{1}{\varepsilon}\frac{\imath}{2\pi u}\left[\exp(-\imath 2\pi u x\right]_{-\varepsilon/2}^{\varepsilon/2}$$

which gives

$$\frac{1}{\varepsilon}\frac{\imath}{2\pi u}\left[\exp(-\imath\pi\varepsilon u) - \exp(\imath\pi\varepsilon u)\right]$$

which we can then write as:

$$\frac{1}{\varepsilon}\frac{\imath}{2\pi u} - 2\imath\sin(\pi\varepsilon u)$$

which is then just

$$\frac{\sin(\pi\varepsilon u)}{\pi\varepsilon u} = \mathrm{sinc}(\pi\varepsilon u)$$

now we have from question 1, we that $\mathrm{sinc}(0) = 1$, so we have that

$$\lim_{\varepsilon\to 0}\mathcal{F}\left\{\Delta_\varepsilon(x)\right\} = 1$$

as expected.

7.6 Sines and Cosines

Given the shifting property of the δ-function, begin:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

then show that:

$$\mathcal{F}\left\{\delta(x-a)\right\} = \exp(\imath 2\pi a u)$$

Use this, or otherwise, to calculate

$$\mathcal{F}\left\{\cos(x)\right\} \quad \& \quad \mathcal{F}\left\{\sin(x)\right\}$$

Solution

We can write,

$$\mathcal{F}\left\{\delta(x-a)\right\} = \int \delta(x-a)\exp(-\imath 2\pi u x)\mathrm{d}x$$

then the shifting property gives that this is just the value of exp() as $x = a$, so that

$$\mathcal{F}\left\{\delta(x-a)\right\} = \exp(-\imath 2\pi a u)$$

take the inverse Fourier Transform of both sides gives,

$$\mathcal{F}^{-1}\left\{\exp(-\imath 2\pi a u)\right\} = \delta(x-a)$$

noting that the difference between a *forward* and *inverse* Fourier Transform is just a $-$ sign, then,

$$\mathcal{F}\left\{\exp(\imath 2\pi a u)\right\} = \delta(x-a)$$

let $a = 1/2\pi$ and interchange x and u to give,

$$\mathcal{F}\{\exp(\imath x)\} = \delta\left(u - \frac{1}{2\pi}\right)$$

Now, noting that the Fourier Transform is linear, then:

$$\mathcal{F}\{\cos(x)\} = \frac{1}{2}[\mathcal{F}\{\exp(\imath x)\} + \mathcal{F}\{\exp(-\imath x)\}] = \frac{1}{2}\left[\delta\left(u - \frac{1}{2\pi}\right) + \delta\left(u + \frac{1}{2\pi}\right)\right]$$

and similarly,

$$\mathcal{F}\{\sin(x)\} = \frac{1}{2\imath}[\mathcal{F}\{\exp(\imath x)\} - \mathcal{F}\{\exp(-\imath x)\}] = \frac{1}{2\imath}\left[\delta\left(u - \frac{1}{2\pi}\right) - \delta\left(u + \frac{1}{2\pi}\right)\right]$$

so cos() and sin() Fourier transform to give a single frequency, as expected.

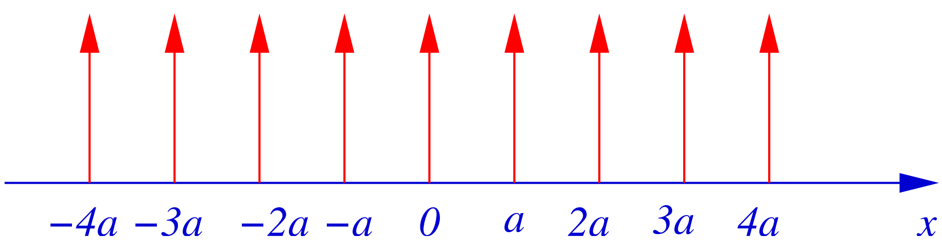
7.7 Comb Function

Calculate the Fourier Transform of a one-dimensional infinte row of delta functions each separated a.

Consider the 3-dimensional case, and compare your result the reciprocal lattice of a simple cubic structure. (this example assumes that you are taking Solid State Physics).

Solution

An infinite row of δ-functions separated by a,



This is known as a δ-Comb, which can be written as

$$\text{Comb}(x) = \sum_{j=-\infty}^{\infty} \delta(x - ja)$$

Note that the Fourier Transform of one δ-function is

$$\mathcal{F}\{\delta(x-a)\} = \exp(-\imath 2\pi a u)$$

so noting that the Fourier Transform is linear, then the FT of the Comb function is

$$F(u) = \sum_{j=-\infty}^{\infty} \exp(-\imath 2\pi j a u)$$

Now we have that,

$$\exp(-\imath 2\pi j a u) = 1 \quad \text{if } 2\pi a u = 2n\pi$$

so that then

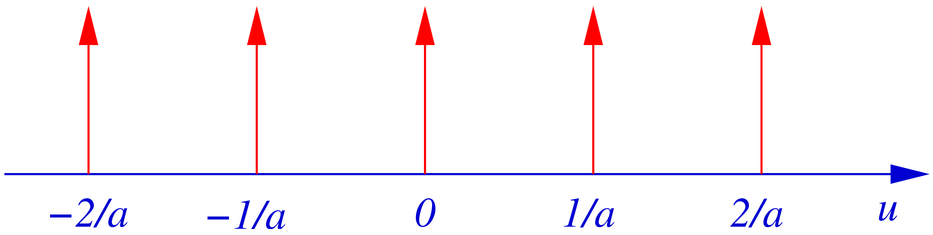
$$u = \frac{n}{a} \quad \Rightarrow \quad \exp(-\imath 2\pi jau) = 1 \quad \forall j$$

to the Fourier Transform

$$\begin{aligned} F(u) &\rightarrow \infty \quad \text{when } u = n/a \text{ (In Phase)} \\ &\rightarrow 0 \quad \text{when } u \neq n/a \text{ (Out of Phase)} \end{aligned}$$

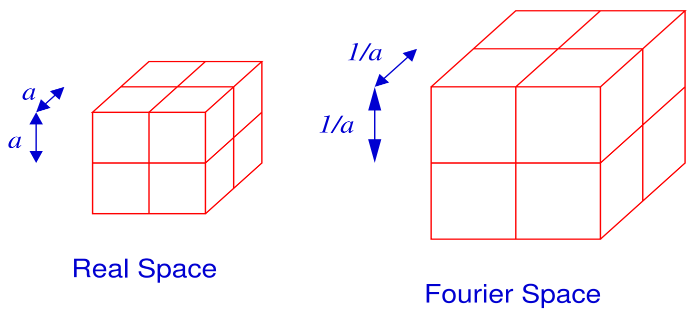
so $F(u)$ is also a δ-Comb with spacing $1/a$, which can be written as

$$F(u) = \sum_{j=-\infty}^{\infty} \delta\left(u - \frac{j}{a}\right)$$



Note the *reciprocal* relation between real and Fourier Space.

In the case of a three-dimensional lattice the atomic locations are given by three-dimensional δ-function (points in three-dimensional space). So a simple cubic lattice is just a three-dimensional δ-Comb. The Fourier transform is separable so we can take the Fourier Transform in each dimension separately. Each transform takes a δ-Comb in real space to a reciprocally spaced δ-Comb is Fourier space, so that the Fourier Transform of a simple cubic lattice is a simple cubic structure in Fourier space.



In solid state physics, the *Reciprocal Lattice* is a Three-Dimensional Fourier Transform of the real space lattice.

All the other lattice structures can be Fourier transformed by considering breaking the structure down into δ-Combs, for example the Fourier transforms of **fcc** is **bcc** etc.

7.8 Convolution Theorm

Prove the Convolution Theorm that if

$$g(x) = f(x) \odot h(x)$$

then we have that

$$G(u) = F(u)H(u)$$

where $F(u) = \mathcal{F}\{f(x)\}$ etc.
The Convolution is frequently described as *Fold-Shift-Multiply-Add*. Explain this be means of sketch diagrams in one-dimension.

Solution

Convolution is defined as

$$g(x) = f(x) \odot h(x) = \int_{-\infty}^{\infty} f(s)\, h(x-s) \mathrm{d}s$$

Now take the Fourier Transform of both sides, to get

$$\int g(x) \exp(-\imath 2\pi u x) \mathrm{d}x = \int \left[\int_{-\infty}^{\infty} f(s)\, h(x-s) \mathrm{d}s \right] \exp(-\imath 2\pi u x) \mathrm{d}x$$

The Fourier Transform is linear, so the order of integration does not matter, so we get

$$G(u) = \int\int f(s)\, h(x-s) \exp(-\imath 2\pi u x) \mathrm{d}s \mathrm{d}x$$

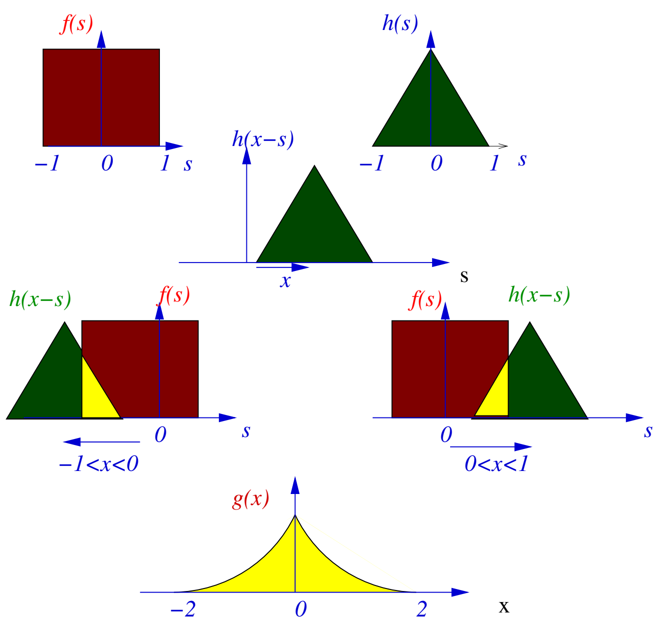
Now let $t = x - s$ so we get

$$\begin{aligned} G(u) &= \int\int f(s)\, h(t) \exp(-\imath 2\pi u(s+t))\, \mathrm{d}s \mathrm{d}t \\ &= \int f(s) \exp(-\imath 2\pi u s) \mathrm{d}s \int h(t) \exp(-\imath 2\pi u t) \mathrm{d}t \\ &= F(u) H(u) \end{aligned}$$

where $F(u) = \mathcal{F}\{f(x)\}$, $H(u) = \mathcal{F}\{h(x)\}$ and $G(u) = \mathcal{F}\{g(x)\}$.
Convolution can be described as:

1. *Fold and Shift* $h(x-s)$ can be interperated at taking function $h(s)$, "Folding" to get $h(-s)$ and "Shifting" by distance x to get $h(x-s)$.
2. *Multiply* the folded and shifted function $h(x-s)$ is $\times$ by $f(s)$
3. *Add* up the area of overlap, (or more formally integrate).

This can be consider diagrammatically as:



7.9 Correlation Theorm

Prove the Correlation Theorm that if

$$c(x) = f(x) \otimes h(x)$$

then

$$C(u) = F(u)H^*(u)$$

and also that

$$h(x) \otimes f(x) = c^*(-x)$$

Show how the Correlation of two images is sometimes called "template-matching".

Solution

Correlation is defined as

$$c(x) = f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(s)\, h^*(s-x)\mathrm{d}s$$

Now take the Fourier Transform of both sides, to get

$$\int c(x)\exp(-\imath 2\pi ux)\mathrm{d}x = \int \left[\int_{-\infty}^{\infty} f(s)\, h^*(s-x)\mathrm{d}s\right]\exp(-\imath 2\pi ux)\mathrm{d}x$$

The Fourier Transform is linear, so the order of integration does not matter, so we get

$$C(u) = \iint f(s)\, h^*(s-x)\exp(-\imath 2\pi ux)\mathrm{d}s\mathrm{d}x$$

Now let $t = s - x$ so we get

$$\begin{aligned} C(u) &= \iint f(s)\, h^*(t)\exp\left(-\imath 2\pi u(s-t)\right)\mathrm{d}s\mathrm{d}t \\ &= \int f(s)\exp(-\imath 2\pi us)\mathrm{d}s \int h(t)^* \exp(\imath 2\pi ut)\mathrm{d}t \\ &= \int f(s)\exp(-\imath 2\pi us)\mathrm{d}s \left[\int h(t)\exp(-\imath 2\pi ut)\mathrm{d}t\right]^* \\ &= F(u)\, H^*(u) \end{aligned}$$

where $F(u) = \mathcal{F}\{f(x)\}$, $H(u) = \mathcal{F}\{h(x)\}$ and $C(u) = \mathcal{F}\{c(x)\}$.
Correlation is very similar to convolution *except* the second function is not *folded* and the direction of the shift is reversed. We have that,

$$c(x) = \int_{-\infty}^{\infty} f(s)\, h^*(s-x)\mathrm{d}s$$

so taking the complex conjugate, we have that

$$c^*(x) = \int_{-\infty}^{\infty} f^*(s)\, h(s-x)\mathrm{d}s$$

Now let $t = s - x$, so we

$$c^*(x) = \int_{-\infty}^{\infty} f^*(t+x)\, h(t)\mathrm{d}t$$

then letting $y = -x$, we get

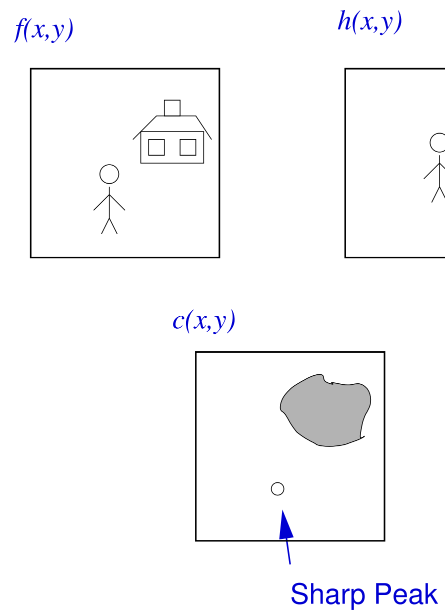
$$c^*(-y) = \int_{-\infty}^{\infty} h(t)\, f^*(t-y)\mathrm{d}t$$

so that we have that

$$h(x) \otimes f(x) = c^*(-x)$$

In most cases the functions $f(x)$ and $h(x)$ will be real, so the complex conjugation does not matter, but the reversal does.

Take a input scene $f(x,y)$ and a target function $h(x,y)$, then the correlation is formed by taking a shifted version of $h(x,y)$, the target, and placing over the input scene $f(x,y)$.



When *man* is located over *house* there is a poor match, so that multiplying and summing the overlap will give a an indistinct *blob*.

When *man* is located over *man* there is a good match so that multiplying and summing will give a sharp *peak*. This height of the *peak* will give the degree at match between the *target* and the *input scene* and the location of the peak will give the location of the *target*. The correlation is thus the degree of match between the target and the input scene, and hence *template-matching*.

7.10 Auto-Correlation

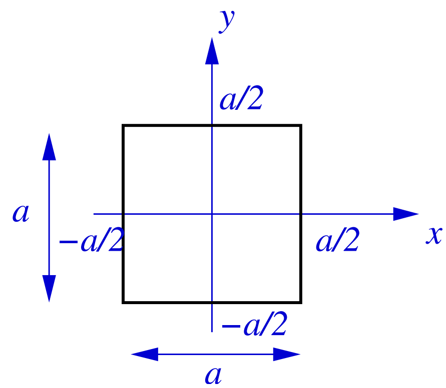
Calculate the *Autocorrelation* of a two-dimensional square of side a centred on the origin. Use *Maple* of *gnuplot* to produce a three-dimensional plot of this function.

Hence calculate the two dimensional Fourier transform of the function

$$\begin{aligned} h(x,y) &= \left(1 - \frac{|x|}{a}\right)\left(1 - \frac{|y|}{b}\right) \quad |x| < a \text{ and } |y| < b \\ &= 0 \quad \text{else} \end{aligned}$$

Solution

Take a square of size $a \times a$ centred about the origin we have



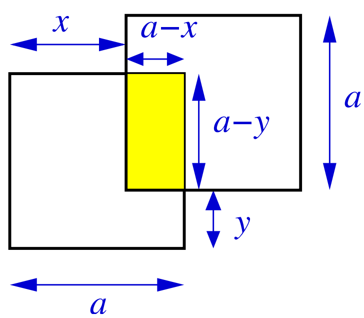
The autocorrelation is given, mathematically, by

$$a(x,y) = \iint f(s,t)\, f^*(s-x, t-y)\,\mathrm{d}s\mathrm{d}t$$

in this case $f(x,y)$ is real.
Physically this means:

1. Shift $f(s,t)$ by amount (x,y).
2. Multiply with the unshifted version.
3. Integrate over the area of overlap.

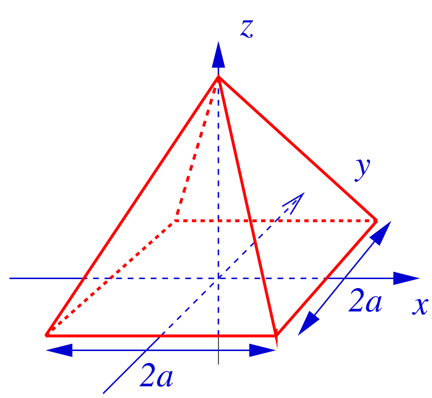
So if we shift by (x,y) we get



so the *Area of Overlap* is

$$a(x,y) = (a-|x|)\,(a-|y|) = a^2\left(1-\frac{|x|}{a}\right)\left(1-\frac{|y|}{a}\right)$$

This is a square pyramid with base $2a \times 2a$



Note that the autocorrelation is **twice** the size of of the original square.

The function $h(x,y)$ is the Normalised Autocorrelation, so that

$$h(x,y) = \frac{a(x,y)}{a^2}$$

The Fourier Transform of $h(x,y)$ is

$$H(u,v) = \mathcal{F}\{h(x,y)\} = \frac{1}{a^2}\mathcal{F}\{a(x,y)\} = \frac{1}{a^2}A(u,v)$$

The autocorrelation theorem gives at

$$\begin{aligned} a(x,y) &= f(x,y) \otimes f(x,y) \\ A(u,v) &= |F(u,v)|^2 \end{aligned}$$

Now $f(x,y)$ is a square of size $a \times a$, so from Question 2 we have that.

$$F(u,v) = a^2 \mathrm{sinc}(\pi a u)\mathrm{sinc}(\pi a v)$$

So we have that

$$A(u,v) = a^4 \mathrm{sinc}^2(\pi a u)\mathrm{sinc}^2(\pi a v)$$

and so the required Fourier Transform

$$H(u,v) = a^2 \mathrm{sinc}^2(\pi a u)\mathrm{sinc}^2(\pi a v)$$

This is much easier than trying to form the direct Fourier Transform.