5 Convolution of Two Functions

The concept of *convolution* is central to Fourier theory and the analysis of Linear Systems. In fact the convolution property is what really makes Fourier methods useful. In one dimension the convolution between two functions, $f(x)$ and $h(x)$ is defined as:

$$g(x) = f(x) \odot h(x) = \int_{-\infty}^{\infty} f(s)\, h(x-s)\, \mathrm{d}s \tag{1}$$

where s is a dummy variable of integration. This operation may be considered the *area of overlap* between the function $f(x)$ and the *spatially reversed version* of the function $h(x)$. The result of the convolution of two simple one dimensional functions is shown in figure 1.

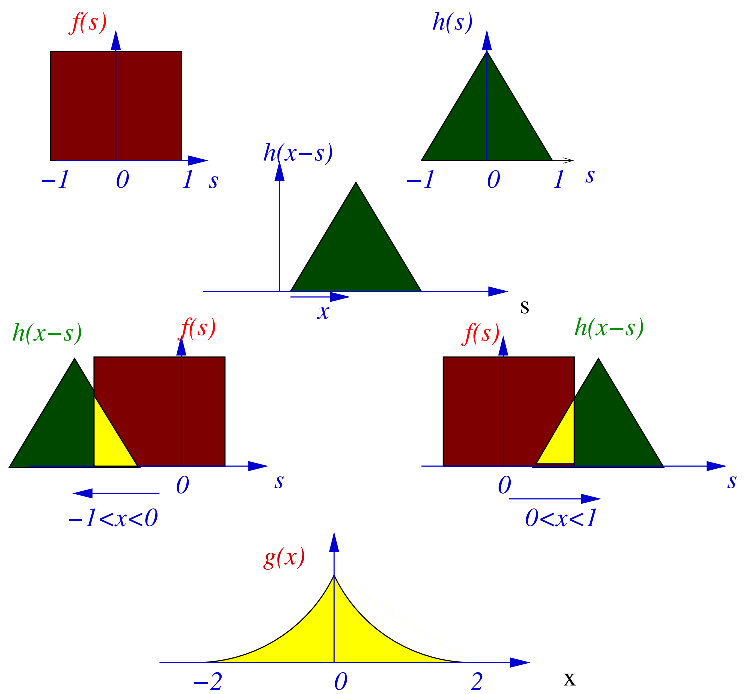

Figure 1: Convolution of two simple functions.

The *Convolution Theorem* relates the convolution between the real space domain to a multiplication in the Fourier domain, and can be written as;

$$G(u) = F(u)\, H(u) \tag{2}$$

where

$$\begin{aligned} G(u) &= \mathcal{F}\{g(x)\} \\ F(u) &= \mathcal{F}\{f(x)\} \\ H(u) &= \mathcal{F}\{h(x)\} \end{aligned}$$

This is the most important result in this booklet and will be used extensively in all three courses. This concept may appear a bit abstract at the moment but there will be extensive illustrations of convolution throughout the courses.

5.1 Simple Properties

The convolution is a linear operation which is distributative, so that for three functions $f(x)$, $g(x)$ and $h(x)$ we have that

$$f(x) \odot (g(x) \odot h(x)) = (f(x) \odot g(x)) \odot h(x) \tag{3}$$

and commutative, so that

$$f(x) \odot h(x) = h(x) \odot f(x) \tag{4}$$

If the two functions $f(x)$ and $h(x)$ are of finite extent, (are zero outwith a finite range of x), then the extent (or *width*) of the convolution $g(x)$ is given by the sum of the widths the two functions. For example if figure 1 both $f(x)$ and $h(x)$ non-zero over the finite range $x = \pm 1$ which the convolution $g(x)$ is non-zero over the range $x = \pm 2$. This property will be used in optical image formation and in the practical implication of convolution filters in digital image processing.

The special case of the convolution of a function with a $\mathrm{Comb}(x)$ function results in replication of the function at the comb spacing as shown in figure 2. Clearly if the extent of the function is less than the comb spacing, as shown in this figure, the replications are separated, while if the the extent of the function is greater than the comb period, overlap of adjacent replications will occur. This operation is central to sampling theory, and image formation and will be discussed in details in the relevant courses. This idea is also central to Solid State Physics where the electron density of a unit cell is convolved with the lattice sites.

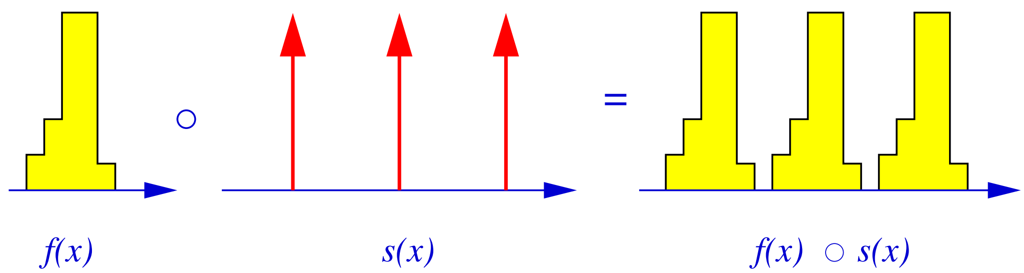

Figure 2: Convolution of function with comb of δ-functions.

5.2 Two Dimensional Convolution

As with Fourier Transform the extension to two-dimensions is simple with,

$$g(x,y) = f(x,y) \odot h(x,y) = \iint f(s,t)\, h(x-s, y-t)\, \mathrm{d}s\, \mathrm{d}t \tag{5}$$

which in the Fourier domain gives the important result that,

$$G(u,v) = F(u,v)\, H(u,v) \tag{6}$$

This relation is fundamental to both optics and image processing and will be used extensively in the both courses.

The most important implication of the *Convolution Theorem* is that,

Multiplication in Real Space	$\Longleftrightarrow$	Convolution in Fourier Space
Convolution in Real Space	$\Longleftrightarrow$	Multiplication in Fourier Space

which is a **Key Result**.

6 Correlation of Two Functions

A closely related operation to *Convolution* is the operation of *Correlation* of two functions. In *Correlation* two function are shifted and the area of overlap formed by integration, but this time *without* the spatial reversal involved in *convolution*. The *Correlation* between two function $f(x)$ and $h(x)$ is given by

$$c(x) = f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(s)\, h^*(s-x)\, \mathrm{d}s \tag{7}$$

where $h^*(x)$ is the *complex conjugate* of $h(x)$[1]. This operation is shown for two simple functions in figure 3. Comparison between the *convolution* in figure 1 and the *correlation* shown that the only difference is that the second function is *not* spatially reversed and the direction of the shift is changed.

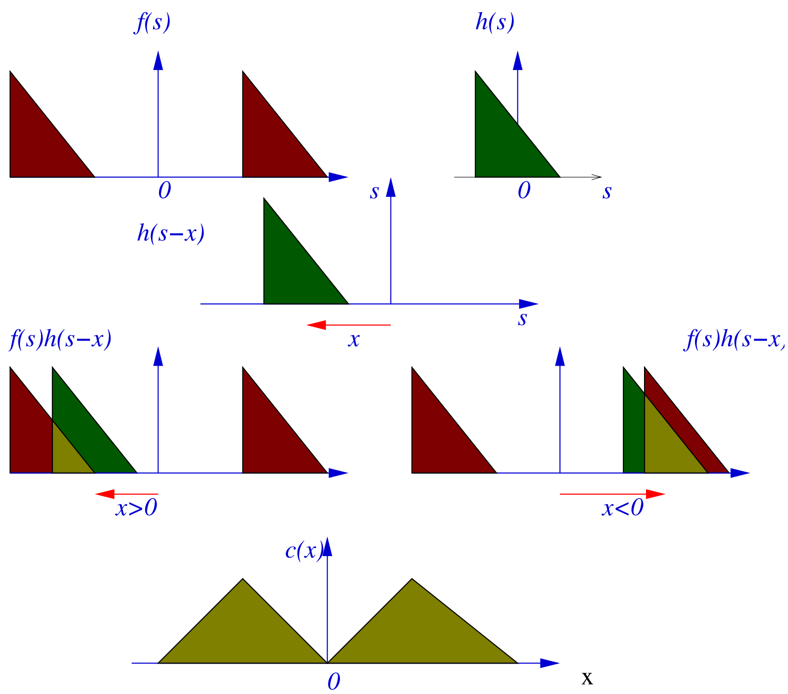

Figure 3: Correlation of two simple functions.

Of more importance, if we consider $f(x)$ to be the "signal" and $h(x)$ to be the "target" then we see that the correlation gives a peak where the "signal" matches the "target". This gives the basis of the simples method of target detection[2].

In the Fourier Domain the *Correlation Theorem* becomes

$$C(u) = F(u)\, H^*(u) \tag{8}$$

where

$$C(u) = \mathcal{F}\{c(x)\}$$

[1] It should be noted that for a real function *complex conjugation* does not effect the function, so if both $f(x)$ and $h(x)$ are *real* then the *Convolution* and *Correlation* differ only by a change of sign, which represents the spatial reversal on one of the functions.

[2] The two-dimensional version of this is considered in question 9.

$$\begin{aligned} F(u) &= \mathcal{F}\{f(x)\} \\ H(u) &= \mathcal{F}\{h(x)\} \end{aligned}$$

It should be noted that the Fourier Transform $H(u)$ is generally complex, and the *complex conjugation is* of vital significance to the operation.

This is again a linear operation, which is distributative, but however is **not** commutative, since if

$$c(x) = f(x) \otimes h(x)$$

then we can show that

$$h(x) \otimes f(x) = c^*(-x)$$

In two dimensions we have the correlation between two functions given by

$$c(x,y) = f(x,y) \otimes h(x,y) = \iint f(s,t)\, h^*(s-x, t-y)\, \mathrm{d}s\, \mathrm{d}t \tag{9}$$

which in Fourier space gives,

$$C(u,v) = F(u,v)\, H^*(u,v) \tag{10}$$

Correlation is used in optics to to characterise the incoherent optical properties of a system and in digital imaging as a measure of the "similarity" between two images.

6.1 Autocorrelation

If we consider the special case of *correlation* with two identical real space functions, we obtain the *correlation* of the input function with itself, being known as the *Autocorrelation*, being,

$$a(x,y) = f(x,y) \otimes f(x,y) \tag{11}$$

so that in Fourier space we have,

$$A(u,v) = F(u,v)\, F^*(u,v) = |F(u,v)|^2 \tag{12}$$

which is the *Power Spectrum* of the function $f(x,y)$. Therefore the *Autocorrelation* of a function is given by the *Inverse Fourier Transform* of the *Power Spectrum*, giving,

$$a(x,y) = \mathcal{F}^{-1}\left\{|F(u,v)|^2\right\} \tag{13}$$

In this case the *correlation* must be commutative, so we have that

$$a^*(-x,-y) = a(x,y)$$

If in addition the function $f(x)$ is real, then clearly the correlation of a real function with it self is real, so that $a(x)$ is real. Therefore for a real function the autocorrelation is symmetric.