1 Introduction

Fourier Transform theory is essential to many areas of physics including acoustics and signal processing, optics and image processing, solid state physics, scattering theory, and the more generally, in the solution of differential equations in applications as diverse as weather modeling to quantum field calculations. The *Fourier Transform* can either be considered as expansion in terms of an orthogonal bases set (sine and cosine), *or* a shift of space from *real space* to *reciprocal space*. Actually these two concepts are mathematically identical although they are often used in very different physical situations.

The aim of this booklet is to cover the Fourier Theory required primarily for the

- Junior Honours course OPTICS.
- Senior Honours course MODERN OPTICS[1] and DIGITAL IMAGE ANALYSIS
- Geoscience MSc course THEORY OF IMAGE PROCESSING.

It also contains examples from acoustics and solid state physics so should be generally useful for these courses. The mathematical results presented in this booklet will be used in the above courses and they are *expected* to be known.

There are a selection of tutorial style questions with full solutions at the back of the booklet. These contain a range of examples and mathematical proofs, some of which are fairly difficult, particularly the parts in *italic*. The mathematical proofs are *not* in themselves an examinal part of the lecture courses, but the results and techniques employed are.

Further details of Fourier Transforms can be found in "*Introduction to the Fourier Transform and its Applications*" by Bracewell and "*Mathematical Methods for Physics and Engineering*" by Riley, Hobson & Bence.

1.1 Notation

Unlike many mathematical field of science, Fourier Transform theory does not have a well defined set of standard notations. The notation maintained throughout will be:

$$\begin{aligned} x,y &\rightarrow \text{Real Space co-ordinates} \\ u,v &\rightarrow \text{Frequency Space co-ordinates} \end{aligned}$$

and lower case functions (*eg* $f(x)$), being a real space function and upper case functions (*eg* $F(u)$), being the corresponding Fourier transform, thus:

$$\begin{aligned} F(u) &= \mathcal{F}\{f(x)\} \\ f(x) &= \mathcal{F}^{-1}\{F(u)\} \end{aligned}$$

where $\mathcal{F}\{\}$ is the Fourier Transform operator.

The character $\imath$ will be used to denote $\sqrt{-1}$, it should be noted that this character differs from the conventional i (or j). This slightly odd convention and is to avoid confusion when the digital version of the Fourier Transform is discussed in some courses since then i and j will be used as summation variables.

[1] not offered in 2006/2007 session.

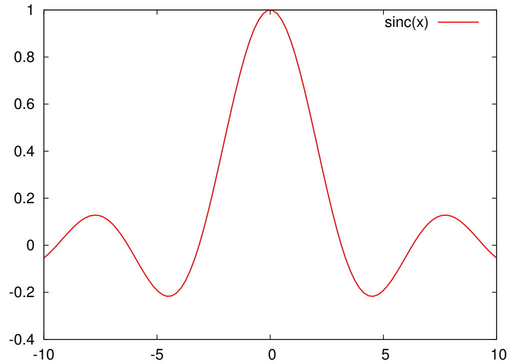

Figure 1: The sinc() function.

Two special functions will also be employed, these being sinc() defined[2] as,

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x} \tag{1}$$

giving $\mathrm{sinc}(0) = 1$[3] and $\mathrm{sinc}(x_0) = 0$ at $x_0 = \pm\pi, \pm 2\pi, \ldots$, as shown in figure 1. The *top hat* function $\Pi(x)$, is given by,

$$\begin{aligned} \Pi(x) &= 1 \quad \text{for } |x| \leq 1/2 \\ &= 0 \quad \text{else} \end{aligned} \tag{2}$$

being a function of unit height and width centered about $x = 0$, and is shown in figure 2

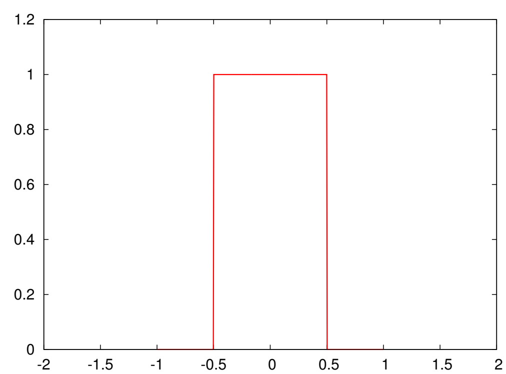

Figure 2: The $\Pi(x)$ function

[2]The sinc() function is sometimes defined with a "stray" 2π, this has the same shape and mathematical properties.

[3]See question 1