Topic: 7 Image Reconstruction

Aim: Lecture covers digital image reconstruction schemes to remove the effect of a imaging point spread function. This includes inverse filtering, Wiener filtering and the non-linear techniques of “clean” and an outline of Maximum Entropy. Geometric image correction will also be discussed as a resampling problem.

Contents:

- Introduction
- Inverse Filtering
- Optimal or Wiener Filter
- CLEAN reconstruction.
- Maximum Entropy Reconstruction
- Geometric Image Correction

Introduction

Aim of *Image Reconstruction* is to remove or compensate for the imaging system aberrations, characterised by PSF $h(i,j)$.

Linear convolution model,

$$g(i,j) = f(i,j) \odot h(i,j) + n(i,j)$$

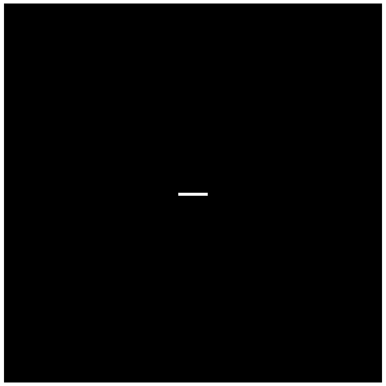
In the initial processes, assume this linear relation and an additive Gaussian noise model.

Assumptions valid for a large range of practical systems.

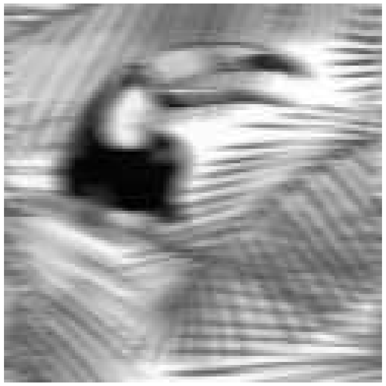
In all system we require to know, or have a good guess for $h(i,j)$ to get a good reconstruction.

Linear Blur Example

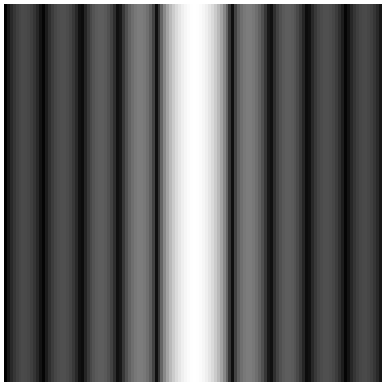
Input image



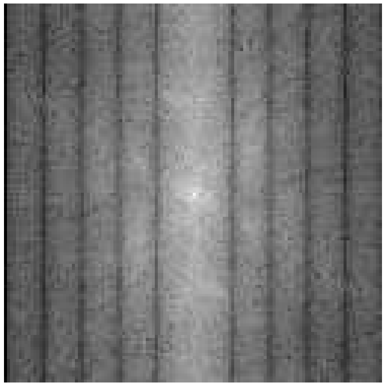
Linear Blur PSF



Blurred Image



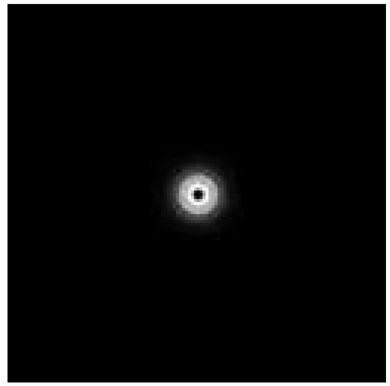
OTF $H(u,v)$



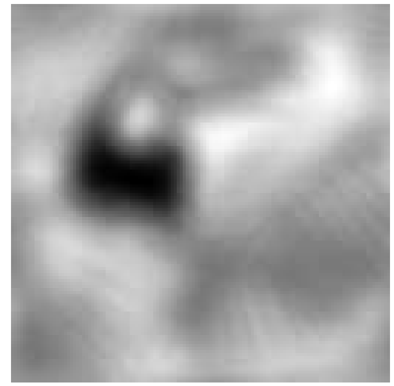
FT Blurred Image

Defocus Example

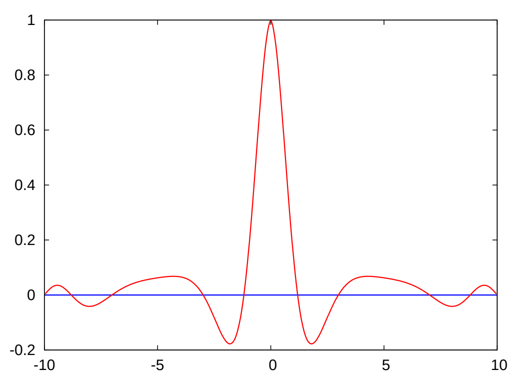
Input image



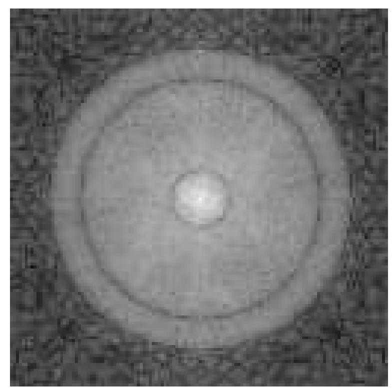
Defocus PSF



Defocused Image



OTF $H(u,v)$



FT Defocused Image

Inverse Filtering

We want to *recover* $f(i,j)$ having *detected* $g(i,j)$:

In Fourier space we have

$$G(k,l) = F(k,l)H(k,l) + N(k,l)$$

if we know (or can calculate) $H(k,l)$, simplest estimate given by

$$\tilde{F}(k,l) = \frac{G(k,l)}{H(k,l)} = F(k,l) + \frac{N(k,l)}{H(k,l)}$$

If $N(k,l) = 0$, exact solution, so problem solved!

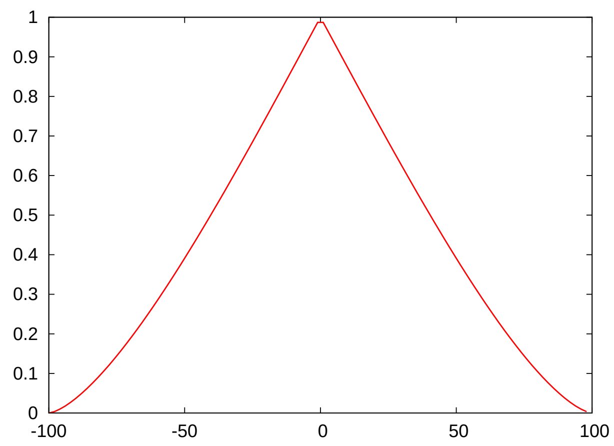
Inverse Filtering I

Major Problems:

Even for tiny amounts of noise, $n(i,j)$ is Gaussian Random Noise, then:

$$\langle |N(k,l)|^2 \rangle \approx \text{constant}$$

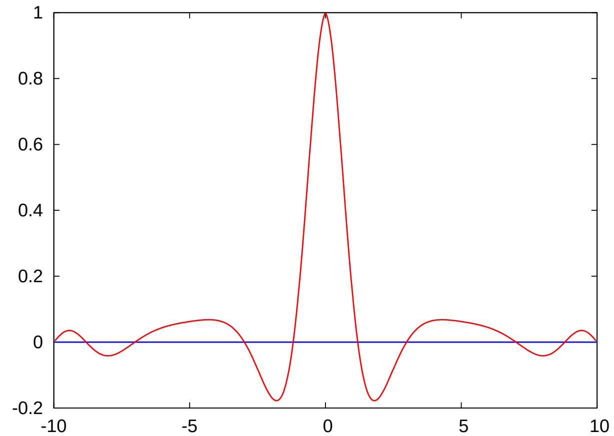
while $H(k,l) \to 0$ at high spatial frequencies.



So noise term will dominate at high frequencies and corrupt the reconstruction.

Inverse Filtering II

All useful situations: (defocus)



Multiple zeros, start a low spatial frequencies.

Modify the inverse filter to

$$\begin{aligned}\tilde{F}(i,j) &= \frac{G(i,j)}{H(i,j)} \quad \text{for } |H(i,j)|^2 > T \\ &= 0 \quad \text{for } |H(i,j)|^2 < T\end{aligned}$$

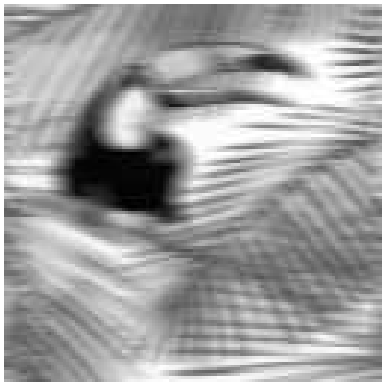
where T is chosen so than $T > |N(i,j)|^2$.

Form reconstuction $\tilde{f}(i,j)$ by inverse FT.

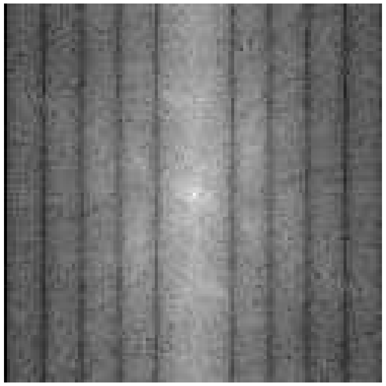
Reconstructions suffer from sharp filter cut-offs and ringing artefacts in reconstruction.

Linear Blur Example

Threshold Inverse Filter:

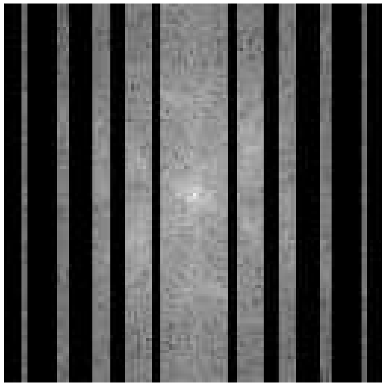


Blurred Image



Fourier Transform

Reconstruction



Fourier Transform

Regions of zero in $\tilde{F}(i, j)$ give rise to ringing in reconstruction $\tilde{f}(i, j)$.

Wiener or Optimal Filter

Reconstruct a *least squares* estimate $\tilde{f}(i,j)$, so that,

$$\langle|\tilde{f}(i,j)-f(i,j)|^2\rangle \quad \text{Minimum}$$

subject to the noise.

Define: an optimal filter $y(i,j)$ such that

$$\tilde{f}(i,j)=g(i,j)\odot y(i,j)$$

We have that:

$$g(i,j)=f(i,j)\odot h(i,j)+n(i,j)$$

so in Fourier space we then have that,

$$G(k,l)=F(k,l)H(k,l)+N(k,l)$$

Therefore by substitution, we have that:

$$\begin{aligned}\tilde{F}(k,l) &= G(k,l)Y(k,l)\\ &= F(k,l)H(k,l)Y(k,l)+Y(k,l)N(k,l)\end{aligned}$$

Fourier Space Minimisation

Since there is *same* information in *Real* and *Fourier* space, we can minimise in Fourier space to give:

$$\langle |\tilde{F}(k,l) - F(k,l)|^2 \rangle \quad \text{Minimum}$$

where $Y(k,l)$ is the minimisation variable.

We therefore have that

$$\frac{\partial}{\partial Y} \langle |\tilde{F} - F|^2 \rangle = 0$$

so that

$$\frac{\partial}{\partial Y} \langle |F - Y H F - Y N|^2 \rangle = 0$$

Noting that the noise is independent and $\langle N \rangle = 0$, we can expand the square and get

$$\frac{\partial}{\partial Y} \langle Y Y^* |W|^2 - Y^* H^* - Y H + 1 \rangle = 0$$

where

$$|W|^2 = |H|^2 + \frac{|N|^2}{|F|^2}$$

Note: Y is complex, so we write $|Y|^2 = Y Y^*$

Fourier Space Minimisation I

by differentiation, we then get

$$\frac{\partial Y^*}{\partial Y}\left\langle \left| Y|W|^2 - H^* \right| \right\rangle + \frac{\partial Y}{\partial Y}\left\langle \left| Y^*|W|^2 - H \right| \right\rangle = 0$$

We note that this is of the form

$$a + a^* = 0$$

so that *both* parts **must** to zero.

If $Y(k,l) \neq \text{Constant}$, then:

$$\frac{\partial Y^*}{\partial Y} \neq 0 \quad \text{and} \quad \frac{\partial Y}{\partial Y} \neq 0$$

so we have the solution that:

$$Y(k,l) = \frac{H^*(k,l)}{|W(k,l)|^2}$$

which can be written as:

$$Y(k,l) = \frac{H^*(k,l)}{|H(k,l)|^2 + \frac{|N(k,l)|^2}{|F(k,l)|^2}}$$

where $||^2$ are the *Power Spectrums*

Estimates for Wiener Filter

This expression gives the optimal filter in terms of

$H(k,l)$	System PSF
$\lvert N(k,l)\rvert^2$	Power spectrum of Noise
$\lvert F(k,l)\rvert^2$	Power spectrum of Ideal image

Point Spead Function: $h(i,j)$ and so $H(k,l)$ is assumed known.

Noise Term: $n(i,j)$ is Guassian Additive noise, so that $|N(k,l)|^2 \approx \text{Constant}$, so we take

$$|N(k,l)|^2 = \sigma_n^2 \quad \text{Variance of Noise}$$

Estimates for Wiener Filter

Power Spectrum: Problem with $|F(k,l)|^2$, (power spectrum of ideal image). Have to make approximation.

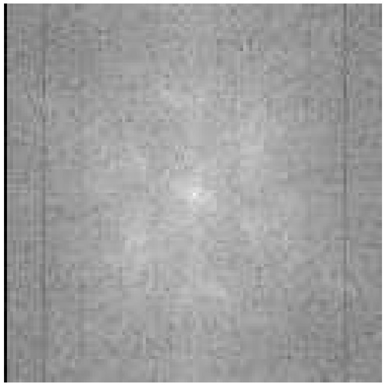
1. Smoothed version of $|G(k,l)|^2$. (Valid if $H(k,l)$ has no zeros).
2. Approximate $|F(k,l)|^2$ by Negative Expotential. (Assumes fractal nature of image, problems close to $(0,0)$).
3. Approximate $|F(k,l)|^2$ by a Gaussian. (Mathematically easy solution.)
4. Take $|F(k,l)|^2 \approx \text{constant}$.

In practice quality of reconstruction only weakly dependent on value of $|F(k,l)|^2$. Frequent the Wiener Filter is written as:

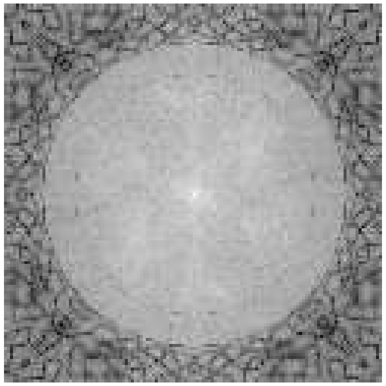
$$Y(u,v) = \frac{H^*(k,l)}{|H(k,l)|^2 + \frac{1}{\mathrm{SNR}^2}}$$

Low Noise Examples

Reconstructions with no added noise and $\mathrm{SNR} = 1000$.



Linear Blur Fourier Transform

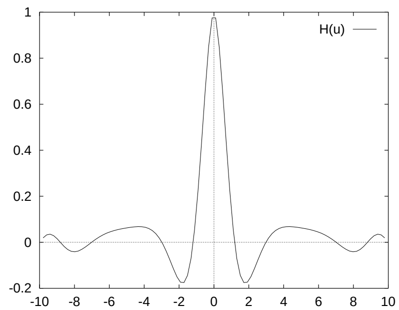


Defoucs Image Fourier Transform

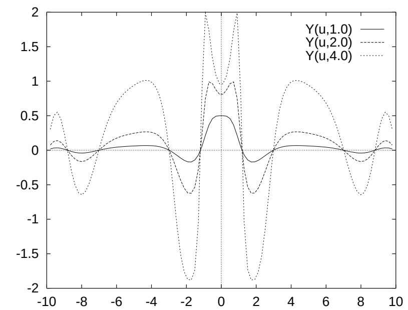
Excellent reconctructions with smooth zero regions in Fourier space.

Effect of SNR

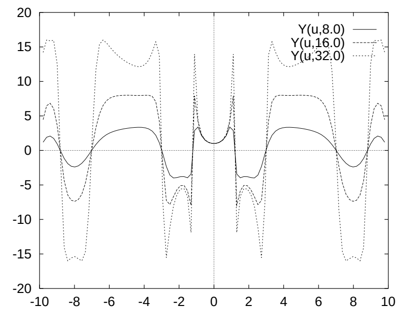
The effect of the SNR term will depend on the shape $H(k,l)$. Look at defocus of a square lens:



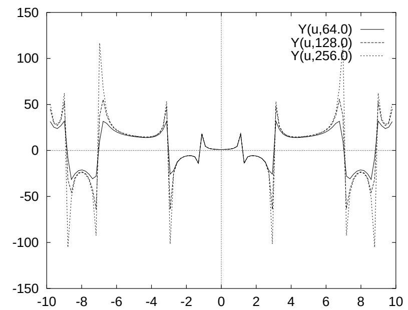
$H(u)$ Defocus



$Y(u)$ SNR $= 1 \rightarrow 4$



$Y(u)$ SNR $= 8 \rightarrow 32$



$Y(u)$ SNR $= 64 \rightarrow 256$

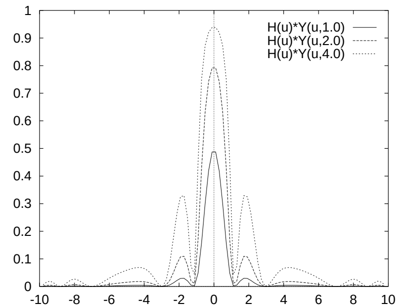
Shape of the filter at high SNR becomes complex, but generally the high the SNR the greater the High Frequency enhancement.

Overall Effect of Reconstruction

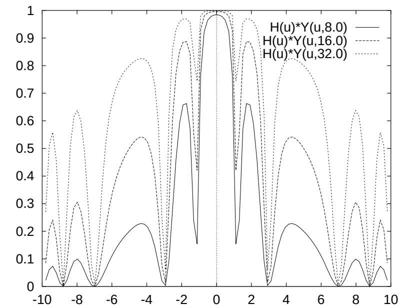
In Fourier space the reconstruction is (without noise),

$$\tilde{F}(k,l) = Y(k,l)\,G(k,l) = (Y(k,l)\,H(k,l))\,F(k,l)$$

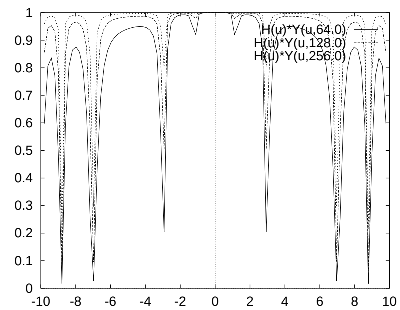
so the overall effect of the blurring followed by the reconstruction is given by $Y(k,l)\,H(k,l)$.



$H(u)Y(u)$ SNR $= 1 \rightarrow 4$



$H(u)Y(u)$ SNR $= 8 \rightarrow 32$



$H(u)Y(u)$ SNR $= 64 \rightarrow 356$

Which shows that at low SNR we get significant Low Pass filtering, while at High SNR we get an almost perfect reconctruction.

Modified Wiener Filter

We have seen that for low(ish) SNR the Wiener Filter acts as a Low Pass filter. Image $f(x,y)$, we have,

$$\frac{\partial f(x,y)}{\partial x} = \mathcal{F}^{-1}\{uF(u,v)\} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = \mathcal{F}^{-1}\{vF(u,v)\}$$

so that

$$|\nabla f(x,y)| = \mathcal{F}^{-1}\{wF(u,v)\} \quad \text{where} \quad w = \sqrt{u^2+v^2}$$

So to enhance edges modify minimisation to

$$\langle|\tilde{F}(u,v) - F(u,v)|^2\rangle + \lambda\langle|w\tilde{F}|\rangle$$

This "can be shown" give,

$$Y(u,v) = \frac{H^*}{|W|^2}\left(\frac{1}{1-\lambda\frac{w^2}{w_0^2}}\right)$$

where w_0 is the bandlimit of the reconstruction system, and λ is range ± 1.

$$\begin{aligned} \lambda \quad &= 0 \quad \text{Unconstrained} \\ &> 0 \quad \text{Edges enhanced} \\ &< 0 \quad \text{Edges reduced} \end{aligned}$$

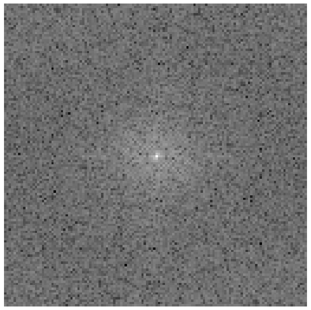
In practical cases the effect of λ will depend in the form of $H(u,v)$.

Clean Algorithm

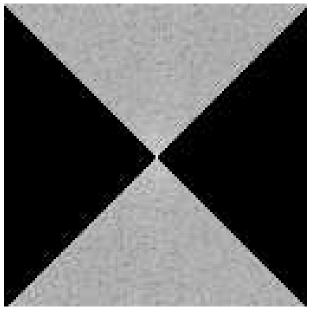
Useful when there are large areas of the Fourier space missing, such as found in Tomography or Radio Astronomy.

Example:

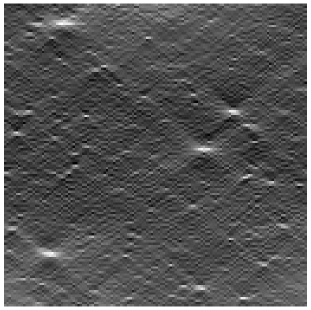
Star Image



Fourier Transform



Collect FT space

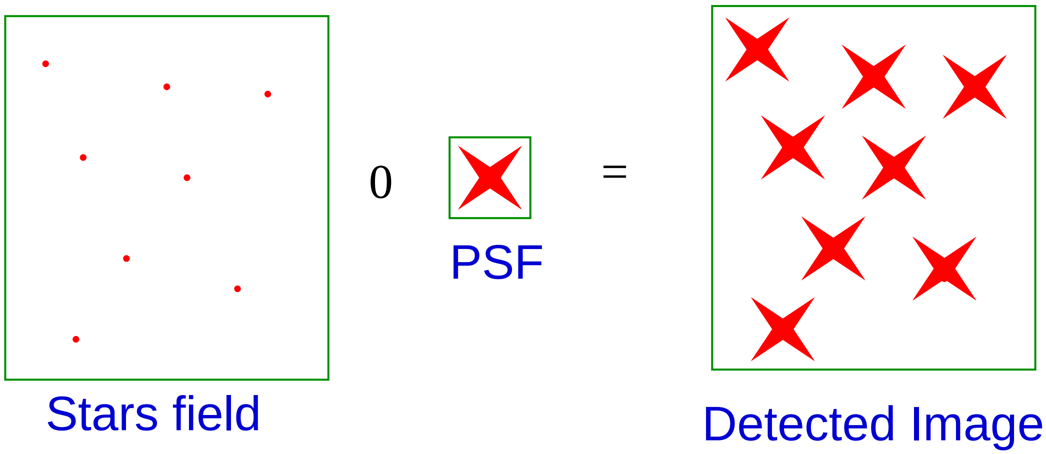


Collected Image

Here the Fourier plane data is *missing* not just scrambled. All linear reconstruction schemes will fail.

Simple Model (for Stars)

Assume collected image is isolated stars convolved with a PSF,



Real space algorithum that searches for PSF in the output and replaces them by stars.

Assume that PSF is sharply peaked in the centre, (good assumption), then scheme is:

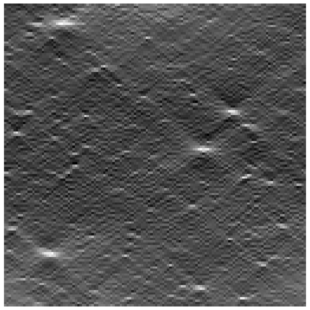
1. Locate Maximum value in image.
2. Record location and height of PSF.
3. Subtract scaled PSF from image at that location.
4. If any peaks left, go to (1)

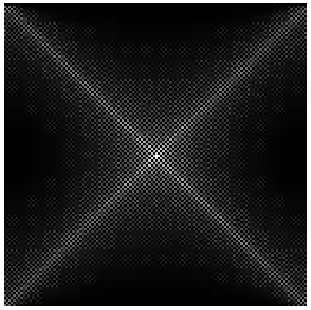
Looks *very* simple, but does it work.

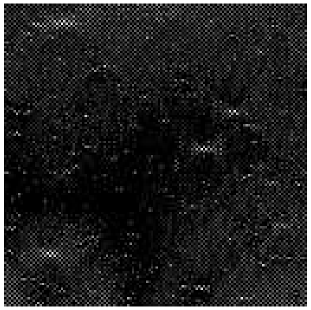
Real Clean Algorithm

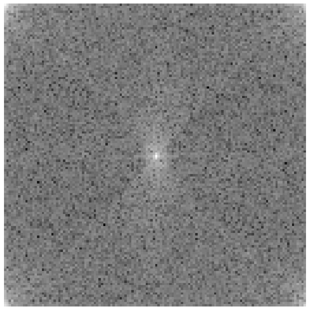
To get to actually work, we need to add,

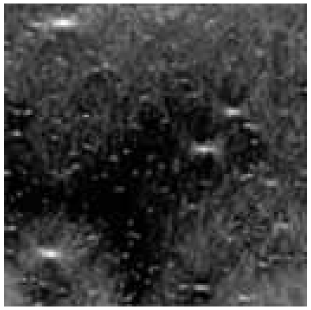
1. Variable scale for removing PSF
2. Care in stopping algorithm.


Collected Image


PSF (stretched)


Reconstruction


FT of Reconstction


Guassian Lowpass

Frequenty Guassian Low Pass filter to smooth the reconstruction.

Maximum Entropy

Maximise *entropy* of reconstruction subject to certain constraints. Produces the smoothest image consistent with the observed data.

Definition of Entropy,

$$H_f = -\langle p(i,j) \log p(i,j) \rangle$$

where

$$p(i,j) = \frac{f(i,j)}{N^2 \langle f(i,j) \rangle}$$

which can be considered as a *probability* since

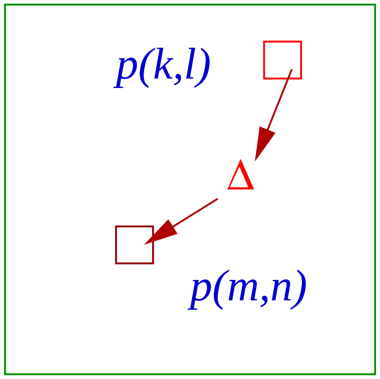
$$\sum_{i=1}^{N} \sum_{j=1}^{N} p(i,j) = 1$$

Maximise H_f subject to the above constraint.

(contraints will make sure that reconstruction is “realistic”)

Why Entropy ?

Consider two pixels $p(k,l)$ & $p(m,n)$,



Move an amount Δ from one to the other, so that

$$\begin{aligned} p(k,l) &\rightarrow p(k,l)-\Delta \\ p(m,n) &\rightarrow p(m,n)+\Delta \end{aligned}$$

Why Entropy ?

we can find the effect of H_f as,

$$H'_f = H_f + \Delta \log\left(\frac{p(k,l)}{p(m,n)}\right)$$

So that

$$H'_f > H_f \quad \text{iff } p(k,l) > p(m,n)$$

so that H_f is a **Maximum** when

$$p(i,j) = \text{constant} = \frac{1}{N^2}$$

which corresponds to the smoothest possible image given the constraints.

Practical Example

In recent work an alternative definition of entropy has be used,

$$H_f = -\langle f(i,j)\ [\log(f(i,j)/A) - 1]\rangle$$

where A is the average brightness or background intensity of the image.

This definition has similar mathematical properties to the above entropy measure with **two** differences.

1. Normalisation constraint removed
2. Free parameter A to characterise image

Normalisation constraint now typically incorporated in constraints on reconstruction.

Max Entropy Deconvolution

Want the smoothest image consistent with the observed data $g(i,j)$. Note also $\log()$ term also forces the reconstruction to be positive.

Image model

$$g(i,j) = h(i,j) \odot f(i,j) + n(i,j)$$

If we have reconstruction $\tilde{f}(i,j)$, then the ideal detected image, must be given by,

$$\tilde{g}(i,j) = h(i,j) \odot \tilde{f}(i,j)$$

so for $\tilde{f}(i,j)$ to be a valid reconstruction, $\tilde{g}(i,j)$ must closely approximate $g(i,j)$. One possible measure is,

$$E = \left\langle \frac{|\tilde{g}(i,j) - g(i,j)|^2}{\sigma_n^2} \right\rangle$$

where σ_n is Standard Deviation of the noise.

Maximum Entropy found by maximisation of

$$Q(\tilde{f}) = H(\tilde{f}) - \lambda E(\tilde{f})$$

Max Entropy Deconvolution I

This can be shown to be solvable by Steepest decent to give iterative scheme

$$\tilde{f}^{k+1} = \tilde{f}^{k} + A \exp\left[-\frac{2\lambda}{\sigma_n^2} h \odot (\tilde{g}^{k} - g)\right]$$

where

$$\tilde{g}^{k} = h \odot \tilde{f}^{k}$$

Require $h(i,j)$, σ_n, A and f^0; typically taken as $f^0 = A$ a constant.

Example (from Skilling et al. Cambridge)

Computationally very heavy algorithm (2 convolutions per iteration), great care has to be taken to prevent iterations diverging.

In practice algorithm will converged to a "good" solution even if $h(i,j)$ is NOT well known.

Geometric Image Correction

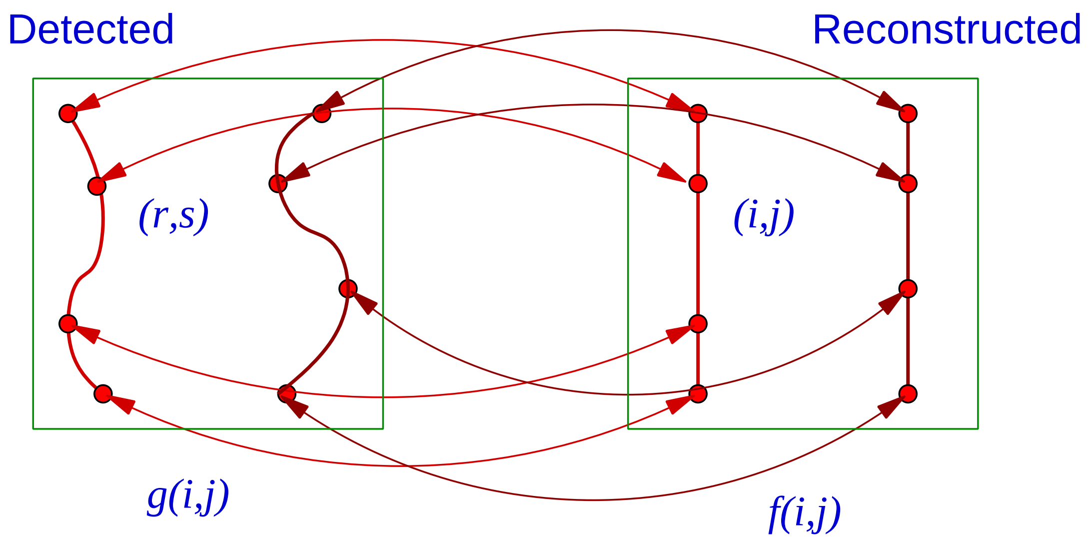
System has now got a space variant PSF; no general solution.

Consider problem as 2-D curve fitting onto a non-linear sampling grid.

For detected image $g(i,j)$ define two distortion functions $r(i,j)$ & $s(i,j)$, such that ideal image is

$$f(i,j) = g(r,s)$$

This formulates the problem as re-sampling $g(i,j)$ on a grid defined by $r(i,j), s(i,j)$



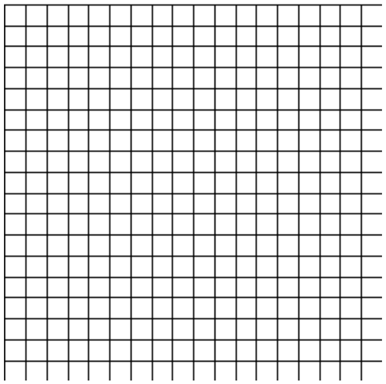
Calculation of Sampling Functions

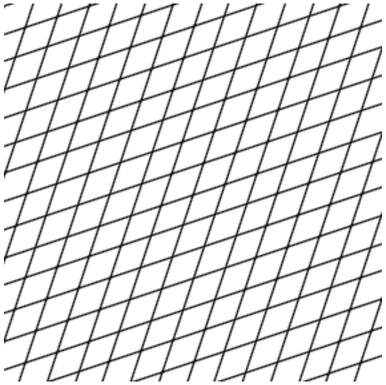
For translation only we have,

$$r = i + a_0$$
$$s = j + b_0$$

More general case of Translation, Scale & Rotation, we need 6 parameters,

$$r = a_0 + a_1 i + a_2 j$$
$$s = b_0 + b_1 i + b_2 j$$





Grid Image Linear Warp

Calculation of Sampling Functions I

Example:

$$a_1 = \cos\theta \qquad b_1 = -\sin\theta$$
$$a_2 = \sin\theta \qquad b_2 = \cos\theta$$

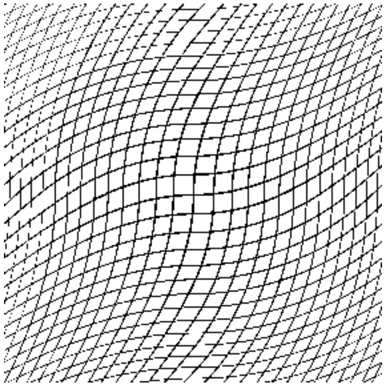
gives a rotation of θ. For $\theta = 30^\circ$,

Calculation of Sampling Functions II

while to correct to include geometric distortions of *skewing* have 12 parameters,

$$r = a_0 + a_1 i + a_2 j + a_3 i^2 + a_4 j^2 + a_5 i j$$
$$s = b_0 + b_1 i + b_2 j + b_3 i^2 + b_4 j^2 + b_5 i j$$

for example:



(This is also used in computer graphics to wrap and image round a three-dimensional object).

For higher order distortions where are 20 parameters.

Calculation of Parameters

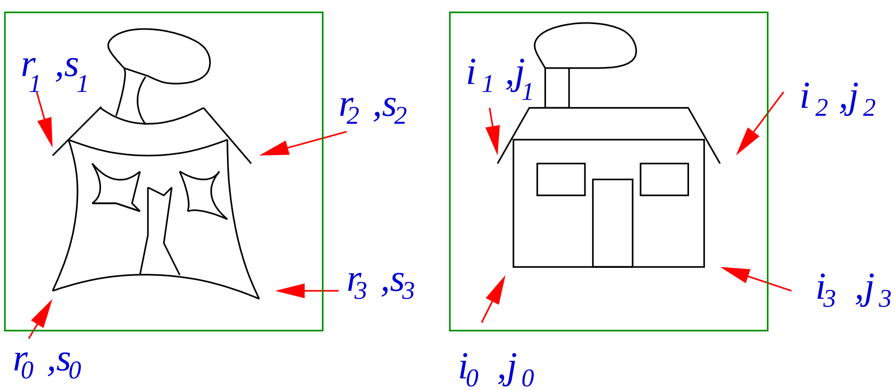
Some cases (video camera), able to calculate parameters from tube design.

Select M *known* features with locations

$$(r_k, s_k) \quad k = 1, \ldots, M$$

while their true locations are at,

$$(i_k, j_k) \quad k = 1, \ldots, M$$



So if the warping parameters are correct then

$$r(i_k, i_k) = r_k \quad s(i_k, j_k) = s_k$$

which is a set of coupled non-linear equations which can be used to caculate the a_i and b_i.

Least Square Error

Better to measure many points on the image and estimate parameters by minimisation of

$$e_a^2 = \sum_{k=1}^{M} \left(r_k - r(i_k, j_k) \right)^2$$

$$e_b^2 = \sum_{k=1}^{M} \left(s_k - s(i_k, j_k) \right)^2$$

Need a *minumum* of 12 points, but usually take more than 100.

Want to spread these poinst over the “important” regions of the image.

This techniques is widely used in satellite data and preparation of images for automatic map making.

Resampling Procedure

We are required to form

$$f(i,j) = g(r(i,j), s(i,j))$$

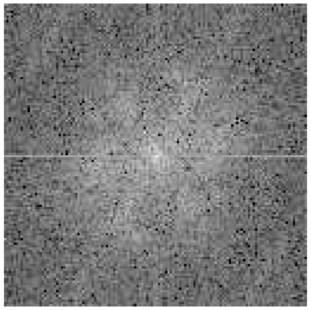
where, in general $r(i,j)$ and $s(i,j)$ will not fall on grid points, so must interpolate between grid points.

Continuous approximation given by

$$g(x,y) = h(x,y) \odot g(i,j)$$

for interpolation fn. $h(x,y)$

Typically either use **zero** or **first** order interpolation, as defined previous, which can result is some resampllig effors



Fourier transform of rotated toucan using **zero** order.

Boundary Effects

In many practical cases values of $r(i,j)$ & $s(i,j)$ may be outside the known range of the image data. **two** solutions.

Cyclic Wrap

As a result of sampling theory,

$$g(N+i, N+j) = g(i,j)$$

although correct from a sampling viewpoint, frequent odd results obtained.

Zero Pad

Take

$$g(r,s) = 0 \; r \text{ or } s \text{ outside image}$$

may give spurious boarder of zero round parts of image, has to be allowed for, especially if then processed by edge detectors.

Summary

In this section we have covered

1. Inverse Filtering
2. Optimal or Wiener Filter
3. CLEAN reconstruction.
4. Maximum Entropy Reconstruction
5. Geometric Image Correction