11 Optical Processing

11.1 Focus of a Laser Beam

A collimated He-Ne laser beam (633 nm) with a Gaussian amplitude of

$$u_0(x,y) = A\exp\left(-\frac{r^2}{r_0^2}\right)$$

is focused by a $\times 40$ microscope objective. Calculate an expression for the amplitude, and intensity distribution in the back focal plane of the objective.

Hint: Assume that the pupil function of the microscope objective is much larger that the laser beam.

If $r_0 = 0.4\,\mathrm{mm}$ calculate the diameter of the input beam and the spot in the back focal plane. For a Gaussian beam the "diameter" is defined by the points that the *intensity* drops to e^{-2}.

Solution

Part a: If pupil function is *much* wider than the beam, then the "effective" pupil function of the lens will be Gaussian, being given by:

$$u_0(x,y) = A\exp\left(-\frac{r^2}{r_0^2}\right)$$

where $r^2 = x^2 + y^2$. The amplitude in the back focal plane is then just the scaled Fourier Transform of this, begin

$$u_2(x,y)\hat{B}_0 \iint A\exp\left(-\frac{(s^2+t^2)}{r_0^2}\right)\exp\left(-\imath\frac{\kappa}{f}(sx+ty)\right)\mathrm{d}s\,\mathrm{d}t$$

where f is the focal length of the objective, in this case $\times 40$ so 4 mm (See solution 1.3).

The Gaussian is seperable (see Fourier Booklet, question/solution 1.3), so we need only consider the 1-D integral,

$$\int \exp\left(-\frac{s^2}{r_0^2}\right)\exp\left(-\frac{\kappa}{f}sx\right)\mathrm{d}s$$

From Fourier Booklet (solution 1.3) we have the standard result that

$$\int_{-\infty}^{\infty}\exp(-bx^2)\exp(iax)\mathrm{d}x = \sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

so with

$$b = \frac{1}{r_0^2} \quad \text{and} \quad a = -\frac{\kappa}{f}x$$

we get the solution to the above integral to be

$$\sqrt{r_0^2\pi}\exp\left(-\frac{\kappa^2 r_0^2}{4f^2}x^2\right)$$

so in two dimensions we get that the amplitude in the back focal plane is

$$u_2(x,y) = \hat{B}_0 A r_0^2 \pi \exp\left(-\frac{r^2}{p_0^2}\right)$$

where

$$p_0 = \frac{f\lambda}{\pi r_0}$$

which is also a Gaussian (as we would expect), with e^{-1} point given by p_0.

Part b: The intensity of the input beam is given by

$$i(x,y) = |u_0(x,u)|^2 = A^2 \exp\left(-\frac{2r^2}{r_0^2}\right)$$

so the e^{-2} point is simply given by $r = r_0$. The diameter of the input beam is thus 0.8 mm (*This is typical of a small He-Ne laser like the ones in the P4 optics laboratory*).

Similarly in the back focal plane the e^{-2} will be given by $r = p_0$, which for $f = 4\,\text{mm}$, and $\lambda = 633\,\text{nm}$, gives a daimeter of $4.03\,\mu\text{m}$.

This result will be used again in the optical processing and spatial filtering lectures.

Bote: if the pupil function is not "much winder" than the Guassian beam we then get a product of the pupil function $p(x,y)$ and the Guassian beam in the pupil which results is the convolution of the focused Guassian and the amplitude PSF of the lens.

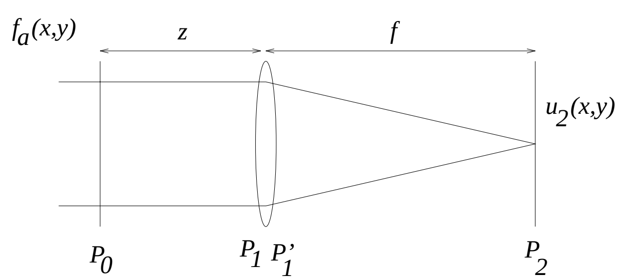
11.2 Fourier Properties of a Lens

Show that if a slide of amplitude transmission $f_a(x,y)$ is illiminated with a coherent collimated beam in the front focal plane of a lens of focal length f, then in the back focal plane the amputide distribution is the scaled Fourier Trasnform of the object.

Hint: First calculate this for the general case of the slide being a distance z in-front of the lens, and then look at the special case of $z = f$.

Solution

Consider the general system with $f_a(x,y)$ a distance z in-front of a lens,



If this is illuminated with a *coherent* beam of unit amplitude, then in plane P_0 we have

$$u_0(x,y) = f_a(x,y)$$

Then in plane P_1 a distance z we get an amplitude

$$u_1(x,y) = u_0(x,y) \odot h(x,y;z)$$

where $h(x,y;z)$ is the *Free Space Propagation Function*. If we assume that we are in the Fresnel region, then we have that

$$h(x,y;z) = -\imath\lambda \frac{\exp(\imath\kappa z)}{z} \exp\left(\imath \frac{\kappa}{2z}(x^2+y^2)\right)$$

so we can write out the full expression for u_1 to be

$$\begin{aligned} u_1(x,y) &= B_0 \exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right) \\ &\quad \int\!\!\int f_a(s,t)\exp\left(\imath\frac{\kappa}{2z}(s^2+t^2)\right)\exp\left(-\imath\frac{\kappa}{z}(xs+yt)\right)\mathrm{d}s\mathrm{d}t \end{aligned}$$

where B_0 is a constant that depends only on z. Now after the lens, in plane P_1 we get,

$$u_1'(x,y) = u_1(x,y)\,p(x,y)\exp(\imath\Phi(x,y))$$

where $p(x,y)$ is the *pupil function* of the lens, and

$$\Phi(x,y) = -\frac{\kappa}{2f}(x^2+y^2)$$

Now if we assume that $p(x,y)$ is *much larger* in extend than $f_a(x,y)$, then we can ignore the *pupil function*, so that

$$u_1'(x,y) = u_1(x,y)\exp\left(-\imath\frac{\kappa}{2f}(x^2+y^2)\right)$$

Finally this amplitude distribution propagates a further distance f to plane P_2, so we get

$$\begin{aligned} u_2(\alpha,\beta) &= u_1'(\alpha,\beta)\odot h(\alpha,\beta;f) \\ &= B_0\exp\left(\imath\frac{\kappa}{2f}(\alpha^2+\beta^2)\right) \\ &\quad \int\!\!\int u_1'(x,y)\exp\left(\imath\frac{\kappa}{2f}(x^2+y^2)\right)\exp\left(-\imath\frac{\kappa}{f}(\alpha x+\beta y)\right)\mathrm{d}x\mathrm{d}y \end{aligned}$$

we can now substitute for $u_1'(x,y)$ which cancels out one of the exponentials under the integral to give,

$$u_2(\alpha,\beta) = B_0\exp\left(\imath\frac{\kappa}{2f}(\alpha^2+\beta^2)\right)\int\!\!\int u_1(x,y)\exp\left(-\imath\frac{\kappa}{f}(\alpha x+\beta y)\right)\mathrm{d}x\mathrm{d}y$$

Now we have to make the final, and messy substitution for $u_1(x,y)$ to get:

$$\begin{aligned} u_2(\alpha,\beta) &= B_0\exp\left(\imath\frac{\kappa}{2f}(\alpha^2+\beta^2)\right) \\ &\quad \int\!\!\int\left[\int\!\!\int f_a(s,t)\exp\left(\imath\frac{\kappa}{2z}(s^2+t^2)\right)\exp(-\imath\frac{\kappa}{z}(xs+yt)\mathrm{d}s\mathrm{d}t\right] \\ &\quad \exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)\exp\left(-\imath(\alpha x+\beta y)\right)\mathrm{d}x\mathrm{d}y \end{aligned}$$

which we can write as:

$$
\begin{aligned}
u_2(\alpha,\beta) &= B_0 \exp\left(\imath\frac{\kappa}{2f}(\alpha^2+\beta^2)\right) \\
&\quad \iiiint \exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)\exp\left(-\imath\kappa\left[\left(\frac{\alpha}{f}+\frac{s}{z}\right)x+\left(\frac{\beta}{f}+\frac{t}{z}\right)y\right]\right)\mathrm{d}x\mathrm{d}y \\
&\quad f_a(s,t)\exp\left(\imath\frac{\kappa}{2z}(s^2+t^2)\right)\mathrm{d}s\mathrm{d}t
\end{aligned}
$$

Look at the central integral of

$$\iint \exp\left(\imath\frac{\kappa}{2z}(x^2+y^2)\right)\exp\left(-\imath\kappa\left[\left(\frac{\alpha}{f}+\frac{s}{z}\right)x+\left(\frac{\beta}{f}+\frac{t}{z}\right)y\right]\right)\mathrm{d}x\mathrm{d}y$$

and we note that this is the Fourier Transform of a Parabolic Phase term, and we are able to solve this. Note that this integral is separable in, so we need only consider the one-dimensional case of:

$$\int \exp\left(\imath\frac{\kappa}{2z}x^2\right)\exp\left(-\imath\kappa\left[\left(\frac{\alpha}{f}+\frac{s}{z}\right)x\right]\right)\mathrm{d}x$$

We now note that we have the identity that,

$$\int \exp\left(-bx^2\right)\exp(\imath ax)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{b}}\exp\left(-\frac{a^2}{4b}\right)$$

So if we let

$$b = -\imath\frac{\kappa}{2z} \quad \& \quad a = -\kappa\left[\left(\frac{\alpha}{f}+\frac{s}{z}\right)x\right]$$

then, after some manipulation, we get that the one-dimensional integral is

$$C_0\exp\left(-\imath\kappa\left[\frac{z}{2f^2}\alpha^2+\frac{1}{2z}s^2+\frac{1}{f}\alpha s\right]\right)$$

where C_0 is a constant that depends only on z. So on Two-Dimensions the central integral becomes

$$C_0\exp\left(-\imath\kappa\left[\frac{z}{2f^2}(\alpha^2+\beta^2)+\frac{1}{2z}(s^2+t^2)+\frac{1}{f}(\alpha s+\beta t)\right]\right)$$

Now if we substitute this back into the expression for u_2, (expressed in terms of x,y), we get, after collection of terms, that

$$u_2(x,y) = D_0\exp\left(\imath\frac{\kappa}{2f}\left(1-\frac{z}{f}\right)(x^2+y^2)\right)\iint f_a(s,t)\exp\left(-\imath\frac{\kappa}{f}(xs+yt)\right)\mathrm{d}s\mathrm{d}t$$

So now if we take the special case of $z = f$, so that the input slide is in the *front focal plane* of the lens, then the quadratic phase term in-front of the integral vanishes, and we get

$$u_2(x,y) = D_0\iint f_a(s,t)\exp\left(-\imath\frac{\kappa}{f}(xs+yt)\right)\mathrm{d}s\mathrm{d}t$$

which is just the scaled Fourier Transform of $f_a(x,y)$, so that, (ignoring the constant D_0), we have that

$$u_2(x,y) = F\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$$

as given in lectures.

Aside 1: We have ignored the pupil function of the lens, this has two effects,

1. In convolves the Fourier Transform with the amplitude PSF of the lens.
2. It limits the size of the input object. It is actually worse than this.

If you but in these terms the algebra get even worse than it is already, anybody want a challenge.

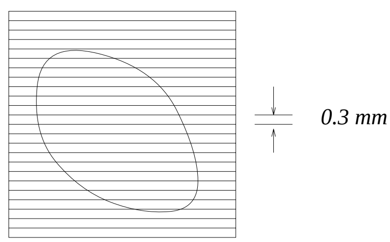
Aside 2: There is an alternative derivation of this result in Goodman Chapter 4, where the initial projection from plane P_0 to P_1 is considered to the a phase shift, but with no diffraction. This gives the right result, but is not a very good physical model.

11.3 Optical Processing

An 4-f optical processing system with 500 mm focal length lenses is used de-stripe lunar photographs as shown in page 472 of *Optics* by Hecht. If the stripes periodic with spacing 0.3 mm sketch the modulus of the Fourier transform of a typical slide and calculate the location of the spots associated with the stripping.

Solution

The image is of type,



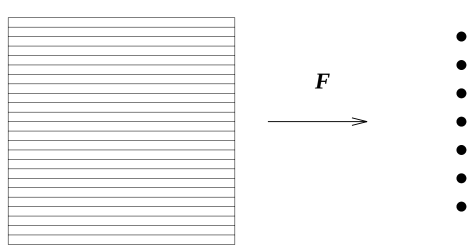
So if we assume that the stripes are represented by the function $s(x,y)$ and the underlying image by $f(x,y)$, then the input image is given by

$$g(x,y) = f(x,y)\, s(x,y)$$

So in Fourier space we get a convolution, so that

$$G(u,v) = F(u,v) \odot S(u,v)$$

where $S(u,v)$ is the Fourier Transform of the stripes,



Mathematically the strips can be written as

$$s(x,y) = \sum_{i=-\infty}^{\infty} \delta(y - i\Delta y)$$

where the spacing is Δy. Noting that this x variable is an effective constant, then from Question 5 in the *Fourier Theory* section the Fourier transform is given by:

$$S(u,v) = \delta(u) \sum_{i=-\infty}^{\infty} \delta\left(v - \frac{i}{\Delta y}\right)$$

so the the $G(u,v)$ consists of a series of replications of $F(u,v)$ separated by a distance $1/\Delta y$ in the v direction.

In an optical system, the Fourier transform is scaled by a factor λf where f is the focal length of the lens. So for $f = 500\,\mathrm{mm}$ and $\lambda = 633\,\mathrm{nm}$, then the separation of the spots will be $1.05\,\mathrm{mm}$.

11.4 Computer Optical Filtering

Experiment with the program `optical_processing` available on the CP laboratory machines to simulate the optical processing of images using a range of filters $H(u,v)$. The programme is located in:

```
wjh/mo4/examples/optical_processing
```

and is supplied with a selection of images in the same directory. These images are taken as the input *intensity* transmittance, from which the input *amplitude* transmittance is calculated by taking the square root. You can view the initial images with

```
xv <imagefile>
```

The supplied images are:

`toucan.pgm`	Image of Toucan
`grating.pgm`	Horizontal grating
`fringe.pgm`	Fringe pattern
`fan.pgm`	Fan image used in defocus dems.

The program will ask you for:

1. *Input image:*

2. *Filter Type:* The options are:

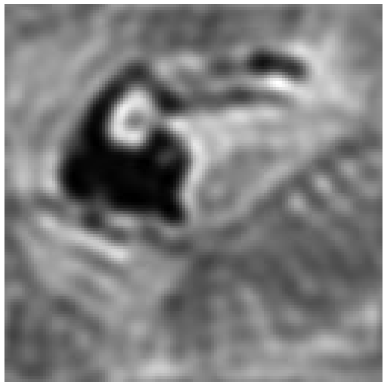
`lowpass`	Lowpass filter
`highpass`	Highpass filter
`bandpass`	Combination of low and high
`guassianlow`	Gaussian lowpass
`gaussianhigh`	Guassian highlass

3. The program will then perform the calculateion and display the output *intensity* via `xv`.

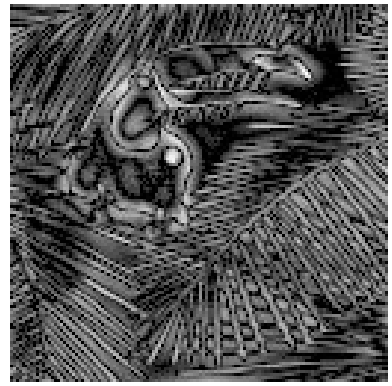
Note: To get sensible images after highpass filtering you may have to use the "Color Edit" window in `xv` to modify the gamma of the output (try 3) which reduces the contrast.

Solution

Here are some example using the (*famous*) `toucan` image,



Lowpass



Highpass

Gaussian Lowpass

Gaussian Highpass

where the the *highpass* images have need enhanced using the *"Color Edit"* facility in `xv`.

Points to note from these images:

1. The Lowpass filtered image is blurred due to removal of high spatial frequencies. It also suffers from sever "ringing" due to convolution with a $\mathrm{J}_1(r)/r$ shape function. This being the $\mathcal{F}\{H(u,v)\}$.

2. The Gaussian Lowpass image is also blurred due to removal of high spatial frequencies, but does not suffer from "ringing" since in this case $\mathcal{F}\{H(u,v)\}$ is a Gaussian which has no secondary maximas.

3. The both Highpass filtered images show the expected "edge enhancement" due to retaining the high spatial frequencies at the expense of the low. Note again the Gaussian highpass has less "ringing" at edges due to the $\mathcal{F}\{H(u,v)\}$ having no secondary maximas.

11.5 Computer Phase Filtering

Experiment with the program `phase_filtering` available on the CP laboratory machines to simulate the optical processing of images using a range of filters $H(u,v)$. The programme is located in:

`wjh/mo4/examples/phase_filtering`

and is supplied with a selection of images in the same directory as listed above.

These images are used to make the "phase only" objects with a user specified maximum phase depth. The resultant phase image is then reconstrcted under either *Darkfield* of *Zernike Phase Contrast* imaging and the *intensity* of the ouput image displayed using `xv`.

The program will ask you for:

1. *Input image:*
2. *Maxiumum phase depth in Wavelengths* (try numbers in the range 0.1 $\rightarrow$ 3.
3. The type of reconstruction. The options are

   `darkfield` or `zernike`

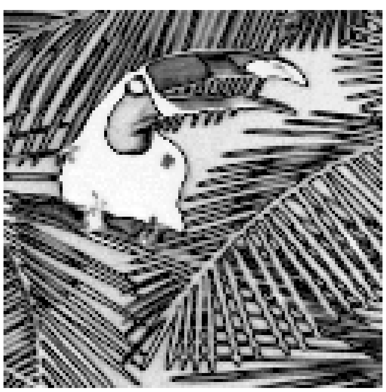
4. The program will then perform the calculation and display the output *intensity* via `xv`.

Again you may have to use the "Color Edit" option to get clear images due to dynamic range problems.

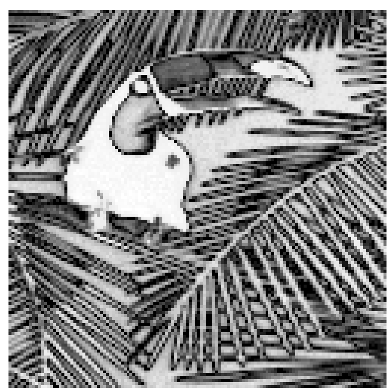
Note: while phase range is small things behave as "expected". However for larger phase ranges things start of go "very wrong" especially for the Zernike reconstructions. Details of this are beyond this course.

Solution

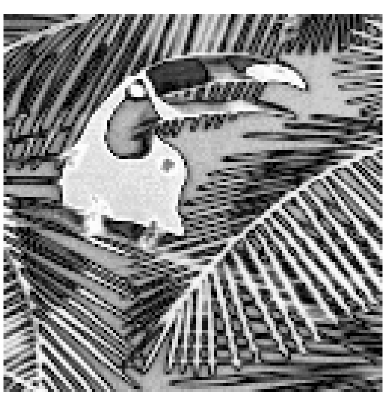
Here are some example using the (*famous*) `toucan` image for for *Darkfiled* reconstructions.



0.1λ phase



0.5λ phase



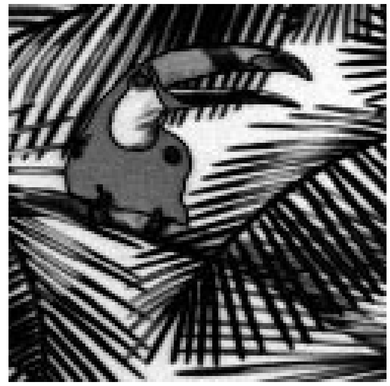
1λ phase 2λ

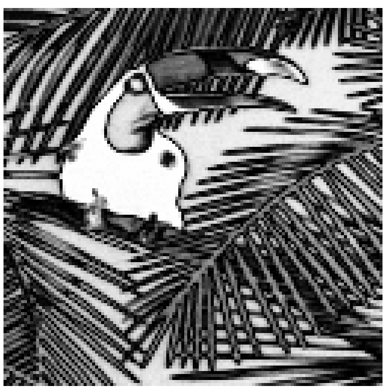
where the the some image have been enhanced *"Color Edit"* facility in xv to make them printable.

These results show that the *darkfield* reconstruction techniques give the expected edge doubling and contrast reversal up to about 1λ of phase variation, but beyond this things start to go severely "wrong". This is actually a better range of phase thickness than would be expected.

The repeat for *Zernike* reconstructions, is shown below,


0.1λ phase


0.5λ phase


1λ phase


2λ

where the the some image have been enhanced *"Color Edit"* facility in xv to make them printable.

Here the results are more surprising with

1. Very small phase (0.1λ) depth we get the "expected" perfect reconstruction with the intensity proportional to the phase depth.
2. At medium phase (0.5λ) we start of see some edge enhancement effects, but still a reasonable image.
3. At large phase ($> 1\lambda$) we get images very similar to the *Darkfield* case with edge doubling and contrast reversals.

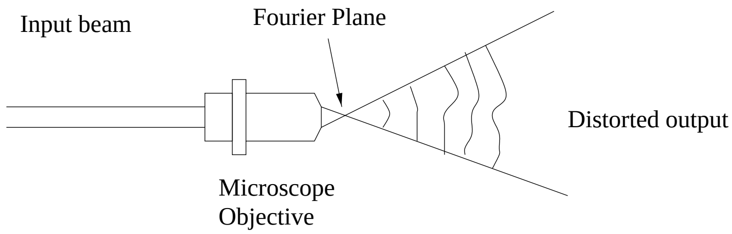
This shown that the *Zernike Phase Contrast* works very well for small phase depth but some very strange effects appears with phase depth of $\approx > 0.5\lambda$. This is as expected since the theory is only valid for *small* phase depths.

11.6 Expanding a laser beam

A collimated laser beam can be expanded into a diverging beam with a short focal length lens, typically a microscope objective. However inperfections in the glass of the objective and dust particles on the lenses result in additional high frequency patterns being superimposed on the exmanding beam. Suggest a scheme for removing this high fequency patterns to give a clean expanding beam.

Solution

The basic system is a collimated beam passing through a convex lens as follows:



This is the same system considered in question 11.1, where if the input beam has a Gaussian amplitude profile with e^{-1} point at r_0, then is the back focal plane (also Fourier Plane) of the objective the amplitude distribution *should* be a Guassian with e^{-1} point at p_0 where

$$p_0 = \frac{f\lambda}{\pi r_0}$$

where f is the focal length of the objective.

The distrortions, inperfections and dust in the objective results in diffraction in the lens that scatters light into high spatial frequencies in the Fourier plane. It is these higher spatial frequencies we want to remove with a filter placed in the Fourier plane.

A Guassian beam has 92% of its energy within a radius given by the e^{-2} intensity radius (which is a also the e^{-1} amplitude radius). So we want to match the filter size of the p_0 radius in the Fourier plane. So we want a filter

$$\begin{aligned} H(x,y) &= 1 \quad \text{for } x^2+y^2 \leq p_0^2 \\ &= 0 \quad \text{else} \end{aligned}$$

so just a "hole" or radius p_0.

This look easy until you start looking at the numbers. For example for the laser and microscope objective detailed in 11.1 the "hole" must be approximately $4\,\mu$m in diameters (1/10th thickness of a human hair!). In practice making the pin-hole is (fairly) easy, this is done by high voltage sparks striking a very thin copper or nickle sheet. Depending on the voltage and film thickness this results in holes of various sizes from $\approx 50\,\mu$m down to about $\approx 1\,\mu$m. These can be purchased from any optical equipment supplier at fairly "modest" cost. The difficult part is that these pin-holes most be positioned *very accurately* in the Fourier plane, typically with a (x,y) accuracy of better than $0.5\,\mu$m and a z accuracy of better than $2\,\mu$m. This requires very accurate mechanical positioners in very stable metal mounts. These are very expensive, heavy and delicate.

Typical system range from the "low cost" system from Ealing Electroptics which contains a 4 mm focal length lens at $5\,\mu$m pinhole at a modest £800 (these are used in the P4 optics labortoratoy), to a "top-of-the-range" automatic system with three piezzo stages and feedback system to optimise the amount of light passed costing rather more than a small Merceedes!

11.7 Fourier Holograms

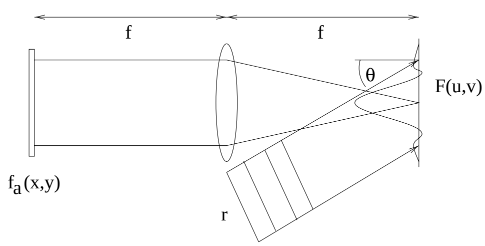
Explain the use of off-axis Fourier holograms in optical correlator system, and derive an expression for the **three** term in the output plane of the system with an input $g(x,y)$ and a hologram recording $F(u,v)$ with a carrier frequency at angle θ.

Calculate the maximum size of input field to prevent overlapping occurring in the output.

Hint: to do this properly is rather difficult since you must consider the extent of the input, correlated with the extent of the object encoded in the Fourier hologram.

Solution

Consider a Fourier plane hologram formed from a slide of amplitude transmission $f_a(x,y)$ with a reference beam at angle θ as shown below.



If the input slide is one focal length infront of the lens, then in the back focal plane the amplitude distribution will be the scaled Fourier Transform of $f_a(x,y)$ given by

$$u_2(x,y) = F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

If the reference beam has amplitude r, then if the two beams are coherent, the *intensity* in the back focal plane is

$$|r\exp(\imath\kappa x\sin\theta) + u_2(x,y)|^2$$

so if we write $u_2(x,y) = |u_2(x,y)|\exp(\imath\phi)$, we get the intensity to be:

$$r^2 + |u_2|^2 + 2r|u_2(x,y)|\cos(\kappa x\sin\theta - \phi)$$

which encodes the *complex* $u_2(x,y)$ as high frequency fringes, so is a hologram then encodes $F(u,v)$ the Fourier Transform of the input $f_a(x,y)$.

In this case $|u_2|^2$ is *definitely not* a constant, since it is the squared modules of the Fourier Transform of $f_a(x,y)$ and it highly peaked about $(0,0)$, so we have to write the intensity as

$$h_0 + h(x,y) + \delta h(x,y)$$

where $h_0 = r^2$, $h(x,y) = |u_2(x,y)|^2$ and $\delta h(x,y = 2r|u_2(x,y)|\cos(\kappa x\sin\theta - \phi)$.

If we expose a holographic plate in this plane, develop it, then its amplitude transmittance will be

$$T_a = K\left(h_0 + h(x,y) + \delta h(x,y)\right)^{-\gamma/2}$$

which again we can write as:

$$T_a = K g_0^{-\gamma/2}\left(1 + \hat{h}(x,y) + \delta\hat{h}(x,y)\right)^{-\gamma/2}$$

where $\hat{h}(x,y) = h(x,y)/h_0$ and $\delta\hat{h}(x,y) = \delta h(x,y)/h_0$. Now expanding this to *first* order we get that:

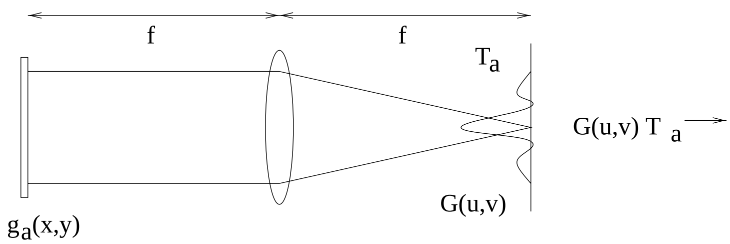
$$\left(1 + \hat{h}(x,y) + \delta\hat{h}(x,y)\right)^{-\gamma/2} \approx 1 - \frac{\gamma}{2}\left(\hat{h}(x,y) + \delta\hat{h}(x,y)\right)$$

so we can then write

$$T_a = T_0 - a\hat{h}(x,y) - a\delta\hat{h}(x,y)$$

were T_0 and a as as given in the slide 7 of the lecture on holography.

If we now place this hologram in the in the optical system below,



with a *second* amplitude slide $g_a(x,y)$ in the front focal plane of the lens. In the back focal plane we get

$$v_2(x,y) = G\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

so the amplitude transmitted through the hologram is

$$v_2(x,y)\,T_a = v_2(x,y)\,T_0 - v_a(x,y)\,a\hat{h}(x,y) - v_a\,a\delta\hat{h}(x,y)$$

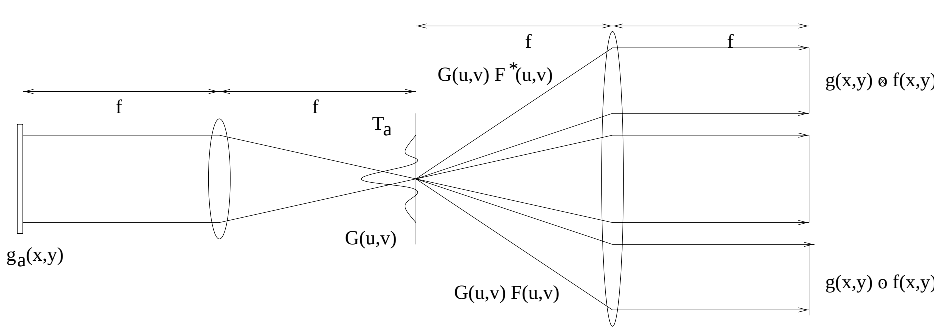
which we can now write in term of $F(u,v)$ and $G(u,v)$ to give, noting that $x = \lambda f u$,

$$T_0\,G(u,v) - a\,G(u,v)\,|F(u,v|^2$$
$$-a\,G(u,v)2r|F(u,v)|\tfrac{1}{2}\left(\exp\left(\imath 2\pi f u \sin\theta - \phi(u,v)\right) + exp\left(-\imath 2\pi f u \sin\theta + \phi(u,v)\right)\right)$$

We can then write $F(u,v) = |F(u,v)\exp(\phi(u,v))$ to get:

$$T_0\,G(u,v) - a\,G(u,v)\,|F(u,v|^2$$
$$-a\,G(u,v)F^*(u,v)\exp(\imath 2\pi f u \sin\theta)$$
$$-a\,G(u,v)F(u,v)\exp(-\imath 2\pi f u \sin\theta)$$

Now let this amplitude distribution fall on a second lens, again one focal length from the hologram as shown below:



Then in the back focal plane of this lens we will form the scaled Fourier Transform of the above amplitude transmission.

As with conventional holography we will get three parts to the reconstruction. If we assume a reversal of coordinates in the output plane we the first term will be, from the correlation and convolution results,

$$T_0\, g_a(x,y) + a\, g_a(x,y) \odot f_a(x,y) \otimes f_a(x,y)$$

which is not useful.

The second term is

$$-a\, g_a(x,y) \otimes f_a(x,y) \odot \delta(x + f \sin\theta)$$

which is the *correlation* of f_a and g_a located about $-f \sin\theta$, which is typically the term we want.

Similarly the third term becomes:

$$-a\, g_a(x,y) \odot f_a(x,y) \odot \delta(x - f \sin\theta)$$

which is the *convolution* of f_a and g_a located about $f \sin\theta$, which is useful, but typically not used.

If θ is large enough, then these three terms will be separated, and we can isolate the $g_a(x,y) \otimes f_a(x,y)$ that we want. Note: we will actually detect $|g_a(x,y) \otimes f_a(x,y|^2$, but provided that both $f_a(x,y)$ and $g_a(x,y)$ are both real and positive (they are simple amplitude transmissions), then the $|\;|^2$ does not present any problems.