Topic 5: Noise in Images
Workshop Solutions

Workshop Questions

5.1 Shape of Poisson Distribution

Use *Maple* or *gnuplot* to plot the Poisson distribution for means of 1,2,4 and 8, and comment on the shapes.

Solution

The Poisson distribution for mean u is given by

$$p_u(n) = \frac{u^n \exp(-u)}{n!}$$

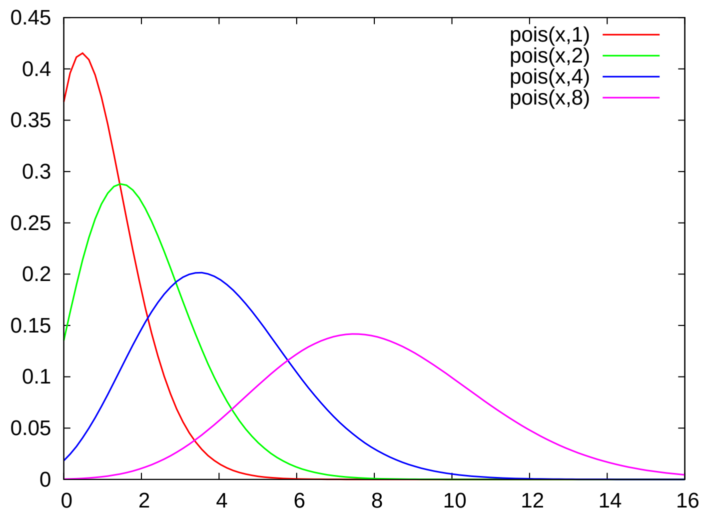
being only defined for n being an integer.

To plot the function using *Maple* or *gnuplot* we need to define a continuous version. Noting that $\Gamma(n+1) = n!$ when n is an integer, we can re-write a continuous version of the Poisson distribution as:

$$p_u(n) = \frac{\exp\left(n\log(u) - u\right)}{\Gamma(n+1)}$$

you also need to rearrange u^n since *gnuplot* only supports raising to integer powers. This is now easily plotted.

The plot of $p_u(n)$ for $u = 1, 2, 4, 8$ is shown below,



This shows that for small u, the Poisson distribution highly asymmetric, but as u increases it become more symmetric in shape and when $u > 8$ begins to *look like* a Gaussian.

5.2 Poisson to Gaussian

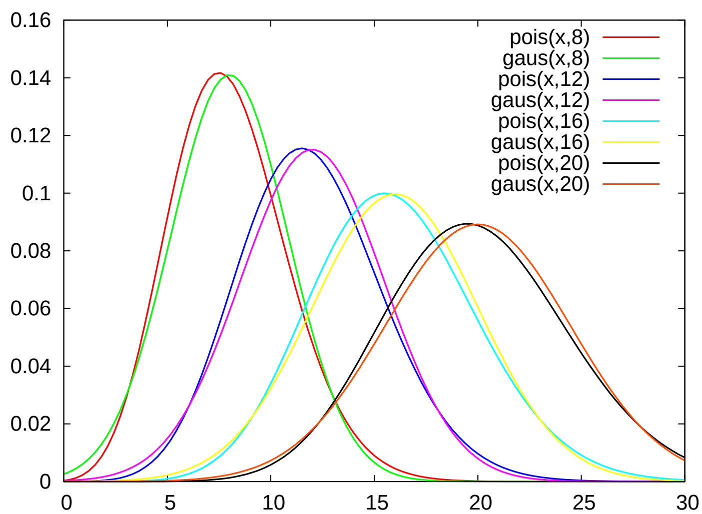
Given that the Poisson distribution can be approximated by a Gaussian of of the form

$$\frac{u^n \exp(-u)}{n!} \rightarrow \frac{1}{\sqrt{2\pi u}} \exp\left(-\frac{(n-u)^2}{2u}\right)$$

when u is large. Plot the Poisson and corresponding Gaussian distribution for a range of u in the range $8 \rightarrow 20$, and comment on the result.

Solution

If we use the continuous expression for the Poisson distribution given in the previous solution plot of the Poisson and the corresponding Gaussian can easily be formed by *gnuplot*. Plot for both distributions for $u = 8, 12, 16, 20$ is shown below:



As u increases the approximation improves. The Poisson is always slightly asymmetric with the maximum slightly before the mean while the Gaussian is always symmetric about the mean, but with $u > 20$ the two distributions are almost identical, the the Gaussian is a good approximation.

5.3 Variance with Noise

If an image is corrupted with signal independent additive noise being given by,

$$f(i, j) = s(i, j) + n(i, j)$$

where $s(i, j)$ is the *true image* or signal, and $n(i, j)$ is the noise then show that:

$$\sigma_f^2 = \sigma_s^2 + \sigma_n^2$$

where σ_f^2, σ_s^2 and σ_n^2 are the variances of the image, signal and noise respectively.

Solution

The additive Gaussian noise has a zero meaned probability density function, so that

$$\langle n(i,j)\rangle = 0 \quad \text{so that} \quad \langle f(i,j)\rangle = \langle s(i,j)\rangle.$$

The noise is also independant of the signal, so in uncorrelated with the signal, so giving

$$\langle s(i,j)\, n(i,j)\rangle = 0$$

The varaince of $f(i,j)$ is, dropping the i,j indices,

$$\sigma_f^2 = \left\langle |f - \langle f\rangle|^2 \right\rangle = \langle |f|^2\rangle - \langle f\rangle^2$$

substitute for f, so give,

$$\begin{aligned} \sigma_f^2 &= \left\langle |s+n|^2 \right\rangle - \langle s+n\rangle^2 \\ &= \left\langle |s|^2 \right\rangle - \langle s\rangle^2 + \langle n\rangle^2 + \langle 2sn\rangle \\ &= \left\langle |s|^2 \right\rangle - \langle s\rangle^2 + \langle n\rangle^2 \\ &= \sigma_s^2 + \sigma_n^2 \end{aligned}$$

which is the relation given in lectures.

5.4 Calculating SNR by Correlation

If you have two images of the same scene taken at different times, show that the SNR can be estimated by

$$\text{SNR} = \sqrt{\frac{c}{1-c}}$$

where c is the *Normalised Correlation* between the two images given by

$$c = \frac{\langle f - \langle f\rangle\rangle \langle g - \langle g\rangle\rangle}{(\langle |f - \langle f\rangle|^2\rangle)^{\frac{1}{2}} (\langle |g - \langle g\rangle|^2\rangle)^{\frac{1}{2}}}$$

where $f(i,j)$ and $g(i,j)$ are the two images.

Solution

Assume that the both images have Guassian additive, signal independant noise, so the two collected images are:

$$\begin{aligned} f(i,j) &= s(i,j) + n(i,j) \\ g(i,j) &= s(i,j) + m(i,j) \end{aligned}$$

where $s(i,j)$ is the signal, or *true image* and $n(i,j)$ and $m(i,j)$ are different realisations of the same noise process.

The noise has a zero meaned Gaussian probability density function, that, dropping the (i,j) we have,

$$\langle n\rangle = \langle m\rangle = 0$$

and also that:

$$\langle |n|^2 \rangle = \langle |m|^2 \rangle = \sigma_n^2$$

where σ_n^2 is the varience of the noise.
The noise is independant of the signal, and *also* the two realisations of the noise are independant, so that,

$$\langle s\,n \rangle = \langle s\,m \rangle = \langle n\,m \rangle = 0$$

Noting that the denominator of c is just the expression for the standard deviations of f and g, the we have that

$$c = \frac{\langle f - \langle f \rangle \rangle \langle g - \langle g \rangle \rangle}{\sqrt{\sigma_f^2}\sqrt{\sigma_g^2}}$$

but noting that $\sigma_f^2 = \sigma_g^2 = \sigma_s^2 + \sigma_n^2$, then the denominator becomes $\sigma_s^2 + \sigma_n^2$.
Substitute $f = s + n$ and $g = s + m$ and expand to get

$$c = \frac{\langle |s|^2 \rangle + \langle s \rangle >^2 + \langle s\,m \rangle + \langle s\,n \rangle + \langle n\,m \rangle}{\sigma_s^2 + \sigma_n^2}$$

all the terms on the numerator containing n or m are zero, so we get that:

$$c = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_n^2}$$

substitute this into the expression dor SNR at get

$$\mathrm{SNR} = \sqrt{\frac{\frac{\sigma_s^2}{\sigma_s^2+\sigma_n^2}}{1 - \frac{\sigma_s^2}{\sigma_s^2+\sigma_n^2}}} = \frac{\sigma_s}{\sigma_n}$$

which is the expression for SNR.

5.5 Variance of Noise

Extend the techniques above for calculating SNR in question 5.4 to give a scheme for calculating the variance of the noise in an image.

Solution

From above we have that the normalsied correlation between two images of the same scene taken at different times is:

$$c = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_n^2}$$

which, noting from above that $\sigma_f^2 = \sigma_s^2 + \sigma_n^2$ then,

$$c = \frac{\sigma_s^2}{\sigma_f^2}$$

in addition, from either image $f(,j)$ *or* $g(i,j)$ we can easily calculate σ_f^2. We can combine these to get:

$$\sigma_f^2(1-c) = \sigma_f^2\left(\frac{\sigma_f^2-\sigma_s^2}{\sigma_f^2}\right) = \sigma_n^2$$

Alternatively: we can calculate

$$\left\langle |f-g|^2 \right\rangle = \left\langle |n-m|^2 \right\rangle = \left\langle |n|^2 \right\rangle + \left\langle |m|^2 \right\rangle = 2\sigma_n^2$$

which is an easier calculation.