Topic 8: Tomographic Imaging

Aim This section covers tomographic imaging and some of its applications mainly related to medical imaging.

Contents:

Introduction

Technique to allow determination of internal structure of object.

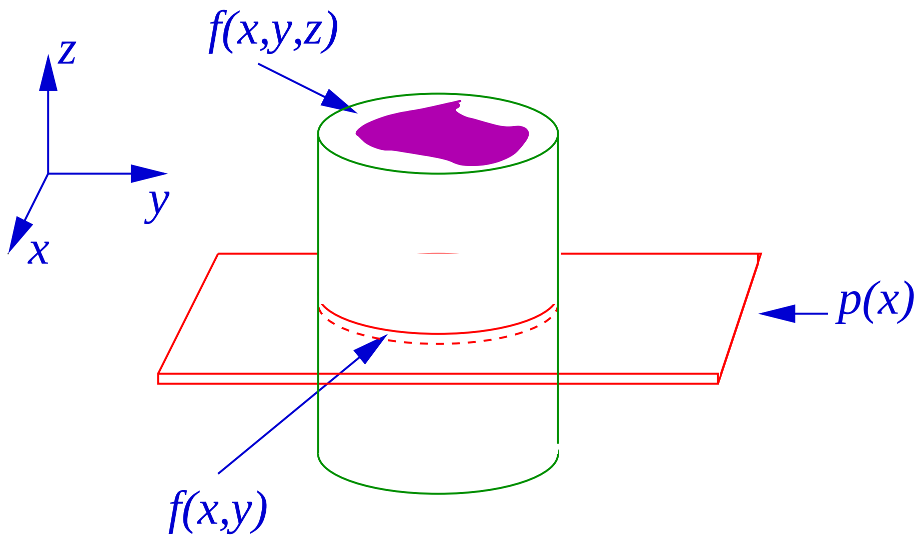
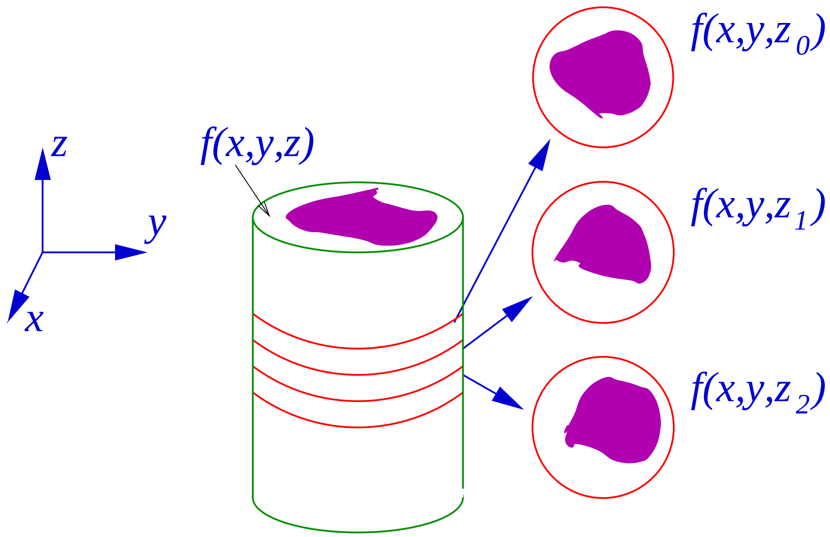


Image gives absorption characteristics of *slice* through object.

Introduction

Full 3-D structure built up from *slices*.



Applications

Medical x-ray: (CT-Scan). X-ray analysis of human body, normally of head, but other sections possible.

Geological/Structural x-ray: X-ray analysis rock samples and fossils. Internal analysis of mechanical components (parts of aero engines)

Astronomical Images: Fan-beam radio astronomy uses same techniques.

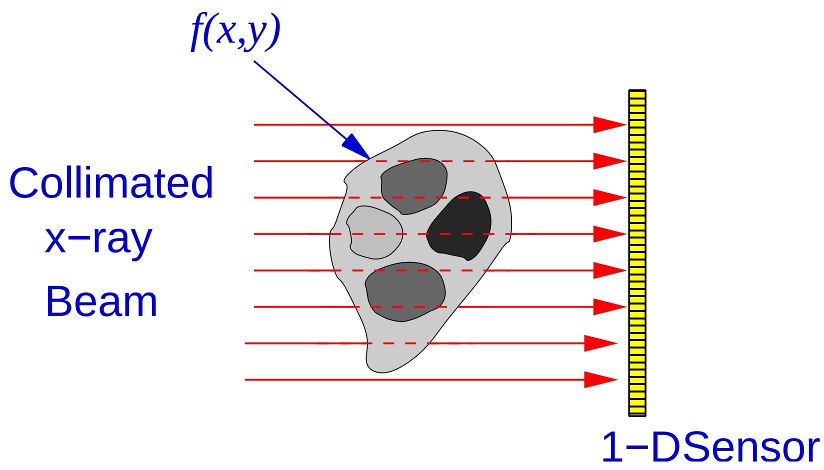
Ultra-sound Tomography: Three-dimensional images from ultra-sound scans.

MRI and PET Medical Imaging: Uses same mathematics of reconstructions from projections.

Characterisation of System

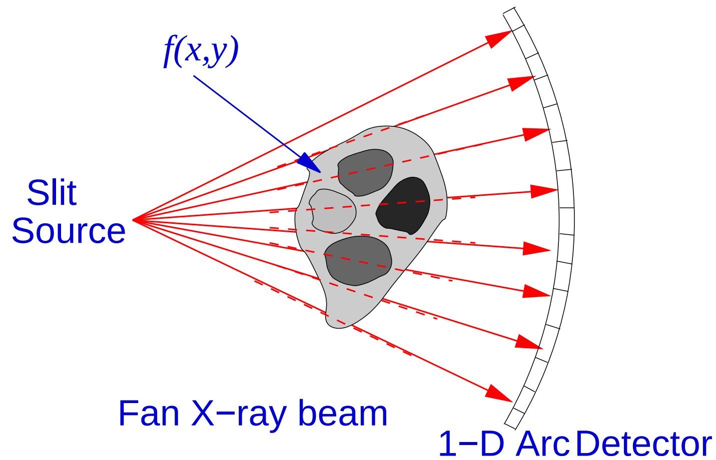
System assumes **NO DIFFRACTION**, two possible geometries,

1) Collimated Beam : Slice of object illuminated by collimated beam with information collected by a linear detector array.



Characterisation of System

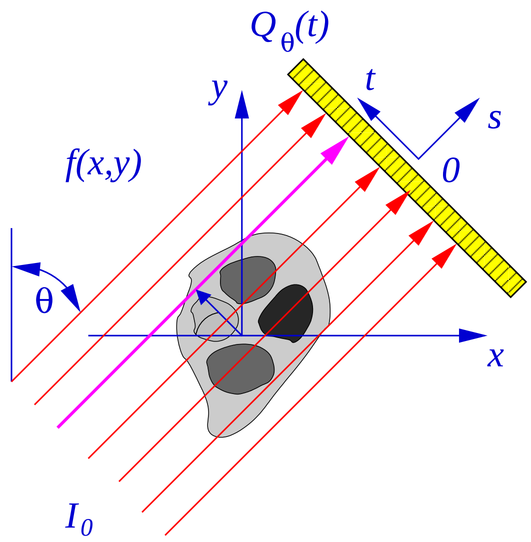
2) Fan Beam : Object illuminated by an expanding beam with linear detector array giving a *fan* section of object.



Assumptions valid for object with absorption variations that are **large** compared to wavelength. eg. X-ray of human body.

Collimated Beam Geometry

Take a thin collimated beam at angle θ of intensity I_0,



2-D slide through object of *absorption* $f(x,y)$, intensity detected at position t from Ray s given by,

$$Q_\theta(t) = I_0\left(1 - \int_{\text{Ray}} f(x,y)\,\mathrm{d}s\right)$$

so normalised absorption of the object is

$$p_\theta(t) = \int_{\text{Ray}} f(x,y)\,\mathrm{d}s$$

Collimated Beam Geometry I

The ray in direction θ that intersects the detector at position t has equation,

$$x\cos\theta + y\sin\theta = t$$

So a line can be represented by

$$\delta(x\cos\theta + y\sin\theta - t)$$

so we have that

$$p_\theta(t) = \int\int f(x,y)\,\delta(x\cos\theta + y\sin\theta - t)\,\mathrm{d}x\mathrm{d}y$$

which can be formed for any θ by, either:

1. Rotating the Object (problem if object is a person!!).
2. Rotating the Detector/Source system.

$p_\theta(t)$ is known as Radon Transform of $f(x,y)$.

The problem: Collect Radon Transform, we want to form $f(x,y)$.

Fourier Inversion Theorem

Take case of $\theta = \frac{\pi}{2}$, so

$$p_{\pi/2}(t) = \iint f(x,y)\,\delta(y-t)\,\mathrm{d}x\mathrm{d}y$$

so, from shifting property, we have,

$$p_{\pi/2}(t) = \int f(x,t)\,\mathrm{d}x$$

Fourier transform of $f(x,y)$ is

$$F(u,v) = \iint f(x,y)\,\exp(-\imath 2\pi(ux+vy))\,\mathrm{d}x\mathrm{d}y$$

so with $u = 0$, we get

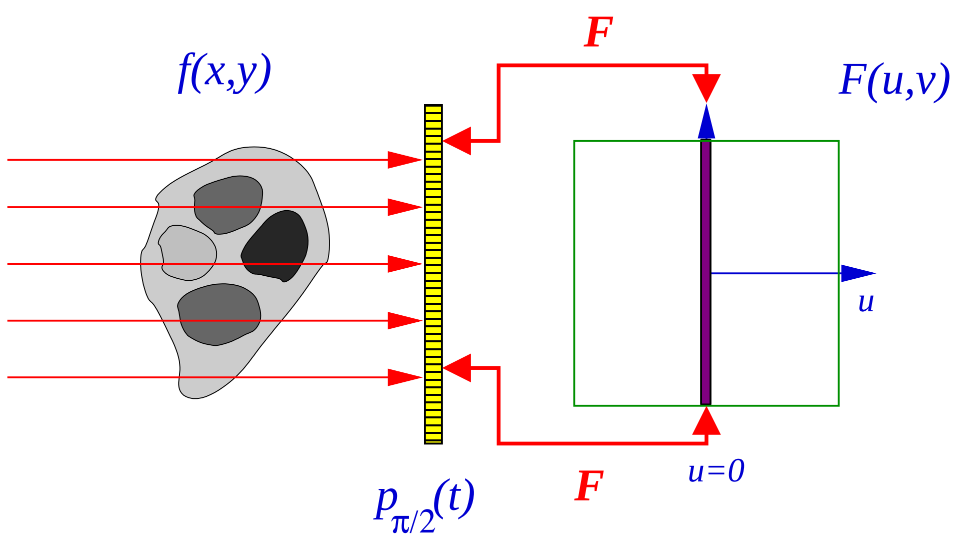
$$F(0,v) = \int \left[\int f(x,y)\,\mathrm{d}x\right] \exp(-\imath 2\pi vy)\,\mathrm{d}y$$

so that

$$F(0,v) = \int p_{\pi/2}(t)\,\exp(-\imath 2\pi vt)\,\mathrm{d}t$$

Fourier Inversion Theorem I

Which says that Fourier transform of the projection forms one-line in Fourier space.



Fourier Inversion Theorem I

Express $F(u,v)$ in polar coordinates,

$$u = w\cos\theta \quad \text{and} \quad v = w\sin\theta$$

to give function $F(w,\theta)$. Also define FT of $p_\theta(t)$ as

$$P_\theta(w) = \int p_\theta(t)\,\exp(\imath 2\pi wt)\,\mathrm{d}t$$

Projection at angle θ then $f(t,s)$ is rotated version of $f(x,y)$ with

$$t = x\cos\theta + y\sin\theta \quad \text{and} \quad s = y\cos\theta - x\sin\theta$$

so that

$$p_\theta(t) = \int f(t,s)ds$$

giving,

$$P_\theta(w) = \int \left[\int f(t,s)\,\mathrm{d}s\right] \exp(-\imath 2\pi wt)\,\mathrm{d}t$$

which in (x,y) coordinates, gives,

$$P_\theta(w) = \int\int f(x,y)\,\exp\left(-\imath 2\pi w(x\cos\theta + y\sin\theta)\right)\,\mathrm{d}x\mathrm{d}y$$

Fourier Inversion Theorem II

which gives that,

$$P_\theta(w) = F(w,\theta) = F(u,v)$$

so by forming

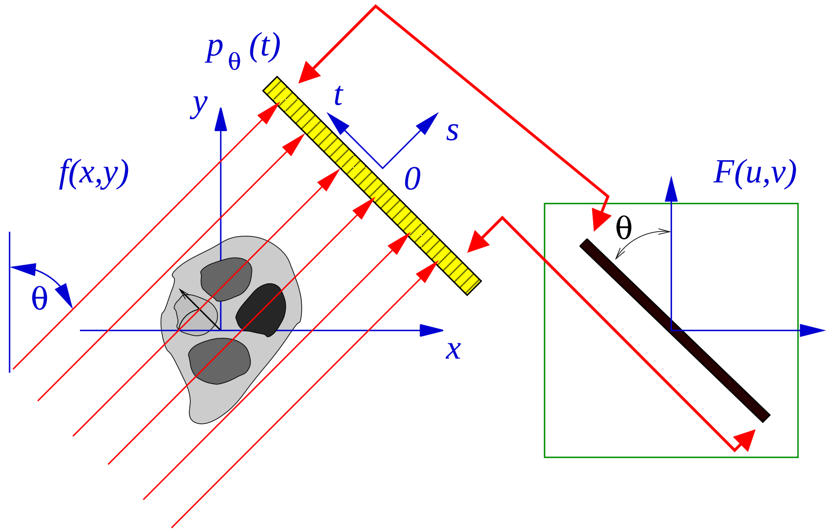
$$p\theta(t) \quad 0 \leq \theta \leq \pi$$

we can form

$$P_\theta(w) \Rightarrow \text{FT of absorption } f(x,y)$$

In terms of Diagrams

Projection $p_\theta(t)$ gives a strip at angle θ in the Fourier Plane,



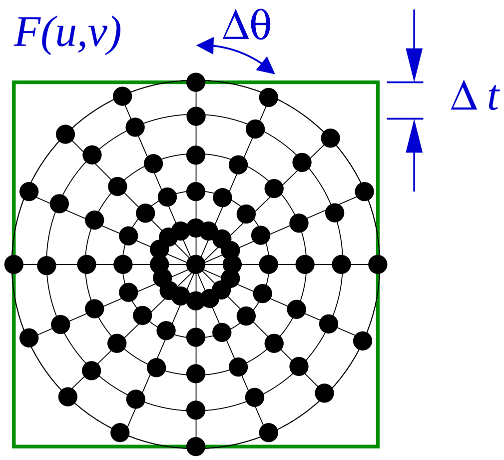
so if we take a whole series of projections at

$$\theta = 0 \Rightarrow \pi$$

then we can fill-in the whole Fourier plane, $F(u,v)$, and then inverse Fourier Transform to get $f(x,y)$.

Interpolation Problem

Practical case : sample $p_\theta(t)$ at Δt, & at $\Delta\theta$. So in Fourier space, sample on **POLAR GRID** of



$$\Delta w = \frac{1}{N\Delta t} \quad \text{and} \quad \Delta\theta$$

To form $f(x,y)$ must re-sample to a Cartesian grid in Fourier space of $\Delta u, \Delta v$ spacing.

Major problem with this method, usually use

1. Zero Order (Nearest neighbour)
2. First Order (Linear interpolation).

Also range of high order techniques, which can help in reducing interpolation noise.

Limited Angle Reconstruction

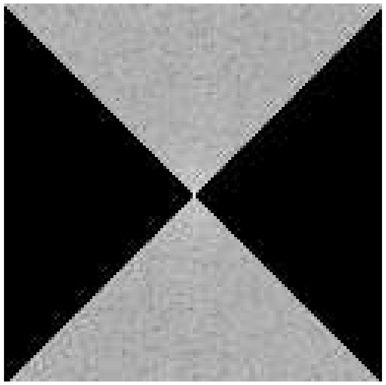
If limited range of angles available, regions of Fourier space will be undefined.

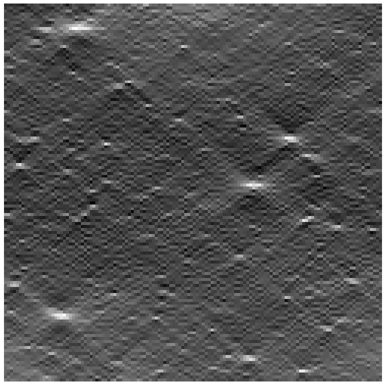
Similar to effect of Fourier filtering, giving a detected Fourier space function,

$$G(u,v) = F(u,v)\,H(u,v)$$

which results in an effective PSF of $h(x,y)$ with the reconstruction given by

$$g(x,y) = f(x,y) \odot h(x,y)$$





Difficult to reconstruct $f(x,y)$ from limited angle data since parts of the Fourier plane are completely missing.

Reasonable results possible in radio astronomy where $f(x,y)$ know to be isolated objects. CLEAN technique possible.

Filtered Back Projection

Based on filtering the projections $p_\theta(t)$ followed by a *back projection*.

In polar coordinates the inverse FT if $F(w,\theta)$ is given by:

$$f(x,y) = \int_0^\infty \int_0^{2\pi} F(w,\theta)\exp\left(\imath 2\pi w(x\cos\theta + y\sin\theta)\right) w \, \mathrm{d}w \mathrm{d}\theta$$

Now if $f(x,y)$ is real, then due to symmetry condition of the FFT, we have that,

$$F(w,\theta+\pi) = F(-w,\theta)$$

we can change the limits of the integration and get:

$$f(x,y) = \int_{-\infty}^\infty \int_0^{\pi} F(w,\theta)\exp\left(\imath 2\pi w(x\cos\theta + y\sin\theta)\right) |w| \mathrm{d}w \, d\theta$$

Consider a rotated coordinate system at angle θ, so that

$$t = x\cos\theta + y\sin\theta$$

we get the simpler expression that:

$$f(x,y) = \int_0^\pi \left[\int_{-\infty}^\infty P_\theta(w)|w|\exp(\imath 2\pi wt)\,\mathrm{d}w\right] \mathrm{d}\theta$$

where we not from previous that

$$P_\theta(w) = F(w,\theta) = F(u,v)$$

where $P_\theta(w)$ is the 1-D Fourier Transform of the projection at angle θ.

Filtered Back Projection I

Now define a filtered projection of

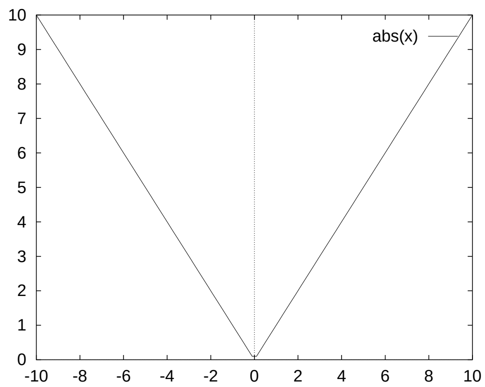
$$q_\theta(t) = \int_{-\infty}^{\infty} P_\theta(w)|w| \exp(\imath 2\pi w t)\, \mathrm{d}w$$

which, from the Convolution Theorem, we have have that:

$$q_\theta(t) = p_\theta(t) \odot h(t)$$

where $h(t)$ is the filter function

$$h(t) = \mathcal{F}\{|w|\}$$

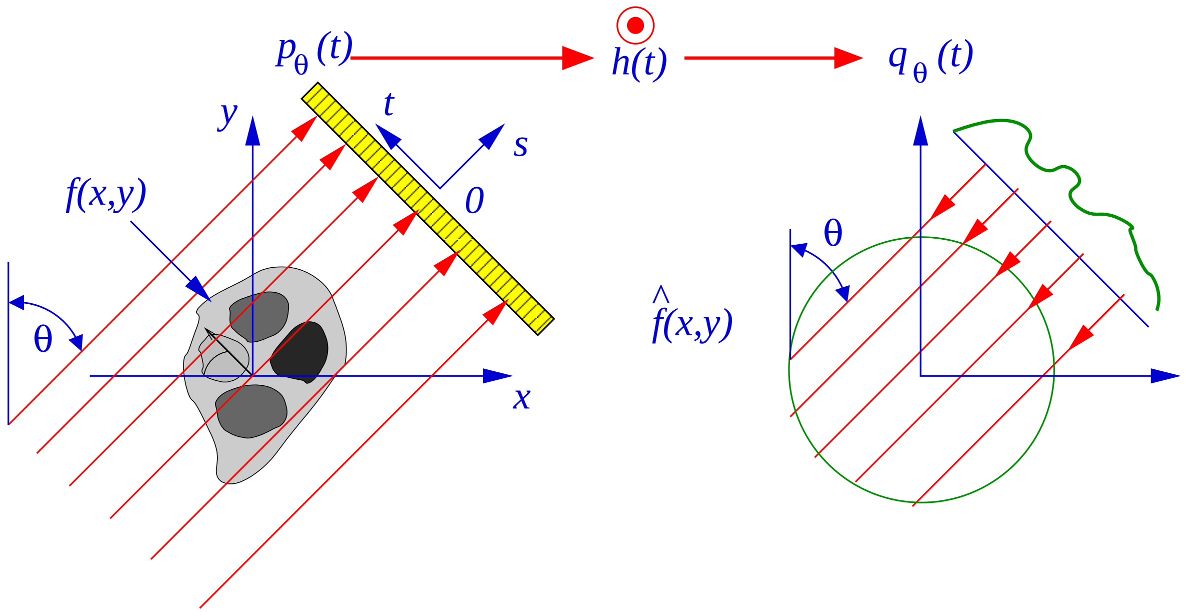


High Pass filtering the 1-D projection. Then final image is given by

$$f(x,y) = \int_0^{\pi} q_\theta(x\cos\theta + y\sin\theta)d\theta$$

Filtered Back Projection I

1. From each angle θ detect $p_\theta(t)$.
2. For each $p_\theta(t)$ form $q_\theta(t)$ by 1-D Convolution.
3. Back-project $q_\theta(t)$ across reconstruction at angle θ.



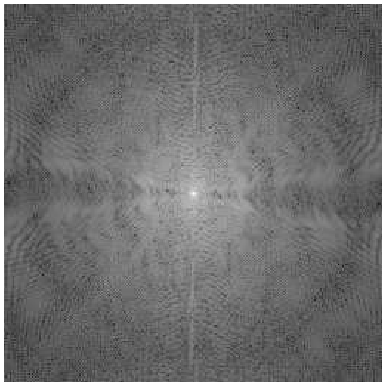
Mathematics of both reconstruction techniques are identical, and get same problem of non-linear sampling in Fourier space.

Why use: Computationally simpler, no final 2-D DFT needed. Not such as issue as it used to be.

Typical Results

Simple CT section of sheeps' neck.

CT Image

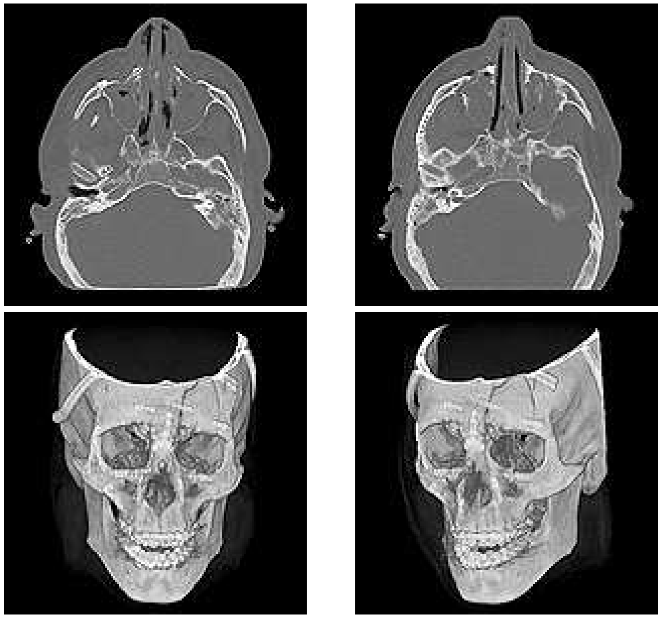


Fourier Transform

CT scan of human head showing injury:

Typical Results

Sections of head formed into full Three-Dimensional model of skull.

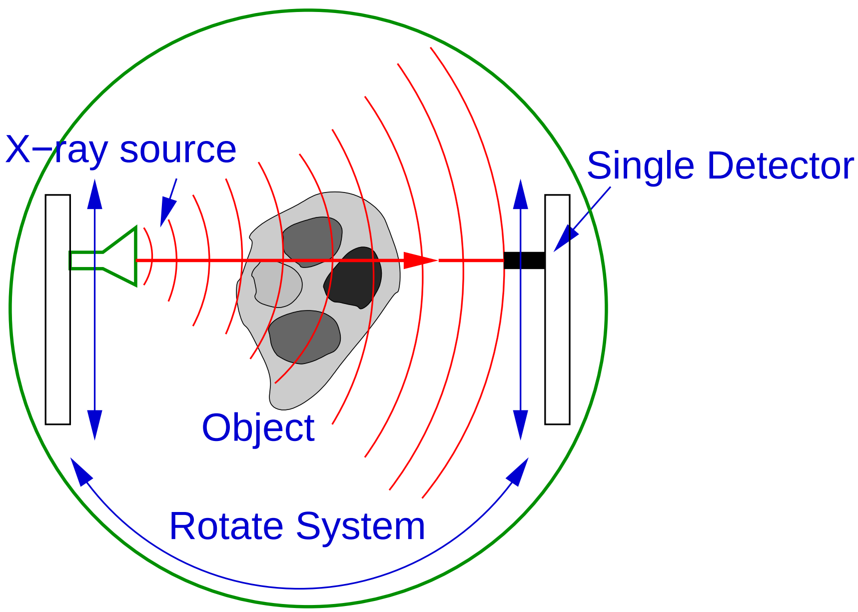


Images from `www.picker.com`

Practical Systems

To implement *Collimated Beam* tomography system we need to produce a thin collimated beam. This in not possible with x-rays (no "lenses" available).

Collimated Beam System: Use single detector, and collect one line projection at a time.



Practical Systems I

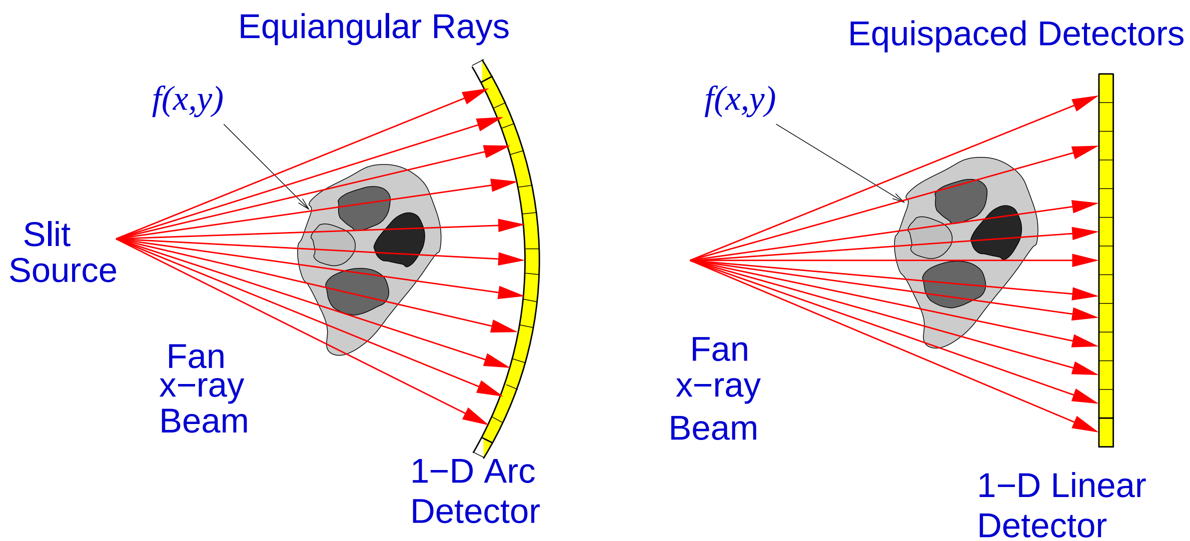
Original system, but

1. Slow to collect data.
2. Most x-rays not used (high does to patient)
3. Mechanically complex (linear and rotation scans)

Basis of system used for rock samples (rotate sample), but not practical for humans.

Fan Beam Reconstruction

Practical system, illuminate with linepoint source and collect data from a linear array detector, hence Fan Beam.



Two Geometries:

Equiangular Rays Detectors equally spaced along arc of circle. Rays at equal angles.

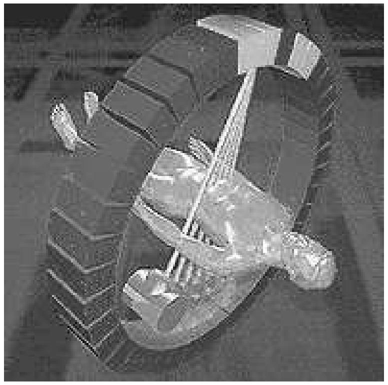
Equispaced Detectors Detectors spaced along linear array. Rays NOT equally spaced.

In practice, all modern machine are of the *Equiangluar Ray* geometry, which is the only one actually used.

Practical Systems

Range of commercial systems, all based on the equiangular rays geometry, schematically

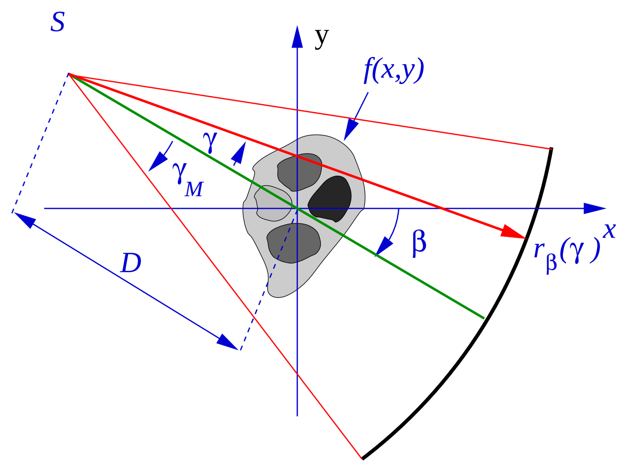
Spiral system by PICKER, which has a 360° ring of detectors and three moving x-ray sources,



Body move through three spiral scans.

Fan beam Reconstruction

Geomtery of fan system is



For a range of β in range $0 \rightarrow 2\pi$ we collect

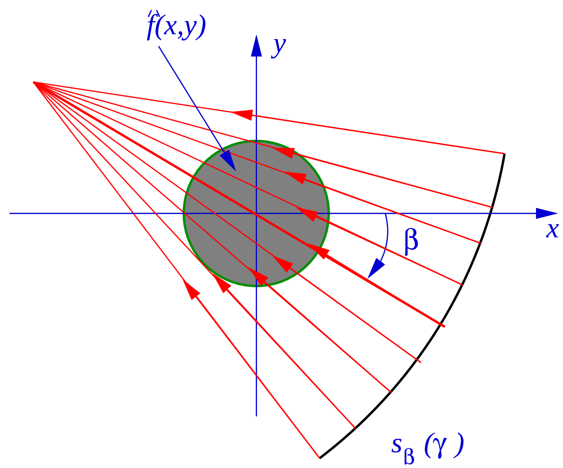
$$r_\beta(\gamma)$$

where γ in the range $-\gamma_M \rightarrow \gamma_M$.

Angular BackProjection

The reconstrcuion can be formulated as a *back-projection* where the colelcted one-dimensional scan is:

1. Scaled by a $\cos()$ of the fan-beam angle.
2. One-dimensional convolution with a filter function.
3. Back-projected along the rays of the equiangulat fan.



So allowing the image to be formed by a series of one-dimensinal Fourier transforms.

Other Techniques

Ray Sorting:
Alternative technique to collect $r_\beta(\gamma)$ and then "sort" the rays by finding the corresponding (t,θ) and thus forming $p_\theta(t)$ (collimated beam), projection. $f(x,y)$ can then be formed by the conventional filtered back projection.

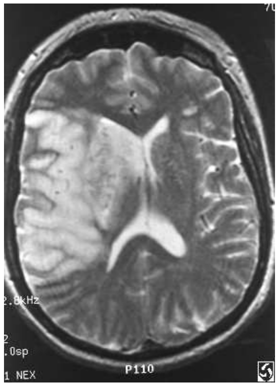
ART: Algebraic Reconstruction Technique that is based on iterative exact inversion of the Radon Transform without using Fourier techniques. Computational very slow, but gives good smooth images.

All detailed in "Computer Tomography, Reconstruction from Projections" by GT Herman, Academic Press 1980.

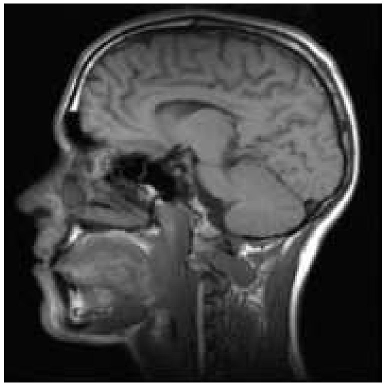
Also computer simulation package called SNARK, which simulated both projections and reconstructions.

Other Tomographic Systems

Magnetic Resonance Imaging: (MRI) Uses nuclear magnetic resonance of the proton at points in the body. The imaged point being selected by varying magnetic and field gradients. Effectively measures water content.



(a) Horizontal Scan



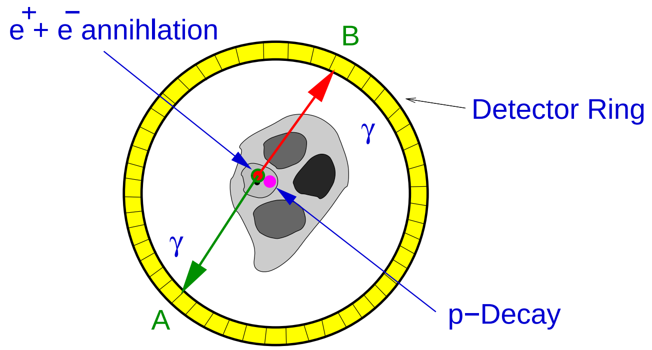
(b) Vertical scan

Clinically very save (no ionising radiation), and very good for “soft tissue” (brain or abdominal), but will not work with bone.

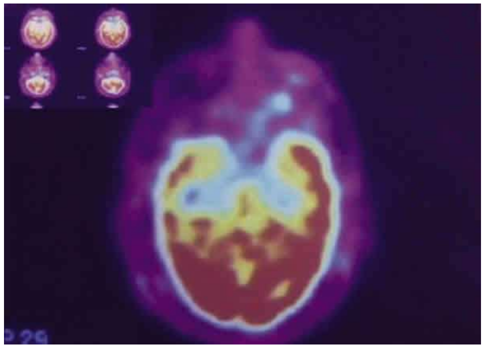
Other Tomographic Systems

Positron Emission Tomography: (PET) Used injected radio isotope (usually ^{15}O), which decays by positron emission (which annihilates with an electron to give 2 γ-rays).

Detect with a ring detector



(a) PET system



(b) Typical image

Gives "line" on which decay occurs. Reconstruction similar to tomography.

Can be made to work in real-time to give live image.

Summary

- Basic concepts of tomographic imaging.
- Collimated beam projection.
- Fourier inversion theorem
- Interpolation problems and issues.
- Filtered back projection for collimated beam reconstruction.
- Pratcical systems and problems.
- Fan beam reconstrcution.
- Other tomographic imaging systems.